

HÖLDER CONTINUITY OF KELLER-SEGEL EQUATIONS OF POROUS MEDIUM TYPE COUPLED TO FLUID EQUATIONS

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ABSTRACT. We consider a coupled system consisting of a degenerate porous medium type of Keller-Segel system and Stokes system modeling the motion of swimming bacteria living in fluid and consuming oxygen. We establish the global existence of weak solutions and Hölder continuous solutions in dimension three, under the assumption that the power of degeneracy is above a certain number depending on given parameter values. To show Hölder continuity of weak solutions, we consider a single degenerate porous medium equation with lower order terms, and via a unified method of proof, we obtain Hölder regularity, which is of independent interest.

1. INTRODUCTION

We study a Keller-Segel model coupled to the fluid equations, where the equation of biological cells is of porous medium type. To be more precise, we consider

$$(1.1) \quad (\text{KS-PME}) \quad \begin{cases} \partial_t n - \Delta n^{1+\alpha} + u \cdot \nabla n = -\nabla \cdot (\chi(c)n^q \nabla c), \\ \partial_t c - \Delta c + u \cdot \nabla c = -\kappa(c)n, \\ \partial_t u - \Delta u + \nabla p = -n \nabla \phi, \\ \nabla \cdot u = 0, \end{cases}$$

where $\alpha > 0$ and $q \geq 1$ are given constants. Here, the unknowns n , c , u and p denote the density of bacteria, the oxygen concentration, the velocity vector of the fluid and the associated pressure, respectively. In addition, the locally bounded functions $\chi : \mathbb{R} \rightarrow \mathbb{R}$ and $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ represent the chemotactic sensitivity and consumption rate of oxygen. Moreover, $\phi = \phi(x)$ is a given potential function. It is known that the above system models the motion of swimming bacteria, so called *Bacillus subtilis*, which live in fluid and consume oxygen. This system has been proposed by Tuval *et al.* in [55] for the case $\alpha = 0$ and $q = 1$, which can be extended to the case $\alpha > 0$ when the diffusion of bacteria is viewed like movement in a porous medium. In this manuscript, we call the above system a Keller-Segel porous medium equation(KS-PME), since fluid equations are restricted to the Stokes system under our considerations.

The main purpose of this paper is to establish the existence of weak and Hölder continuous solutions globally in time for the Cauchy problem of (KS-PME) under general conditions of χ and κ and more extended range of α and q ever known.

We first introduce local Hölder regularity results for a scalar equation under proper conditions on the lower order term, that contributes later obtaining Hölder continuity of a weak solution of system (KS-PME).

In the domain $\Omega_T \subset \mathbb{R}^d \times [0, T]$ for $d \geq 2$, we consider parabolic porous medium type equations in the form of

$$(1.2) \quad n_t = \Delta n^{1+\alpha} + \nabla \cdot (B(x, t)n)$$

for $\alpha \geq 0$ under proper conditions on B where $B : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector field. Roughly speaking, if we are able to obtain regularity results of (1.2) under the condition on B that is expected from $u, \chi(c)$, and ∇c of (KS-PME), then Hölder continuity of a weak solution n of (1.2) yields the same regularity for n , a weak solution of (KS-PME).

In fact, our method of showing Hölder continuity of (1.2) works under the conditions on B and ∇B such that

$$(1.3) \quad B \in L_{\text{loc}}^{2\hat{q}_1, 2\hat{q}_2}(\Omega_T) \quad \text{and} \quad \nabla B \in L_{\text{loc}}^{\hat{q}_1, \hat{q}_2}(\Omega_T)$$

where positive constants $\hat{q}_1, \hat{q}_2 > 1$ satisfy

$$(1.4) \quad \frac{2}{\hat{q}_2} + \frac{d}{\hat{q}_1} = 2 - d\kappa$$

for some $\kappa \in (0, 2/d)$. By letting

$$q_1 = \frac{2\hat{q}_1(1+\kappa)}{\hat{q}_1-1}, \quad q_2 = \frac{2\hat{q}_2(1+\kappa)}{\hat{q}_2-1},$$

the admissible range of constants are obtained from Proposition 2.4 when $p = 2$.

There are many papers working on the continuity of weak solutions to porous medium type equations (refer [15], [3], [5], et al. for a general porous medium equation and special classes of equations). Focusing the main term of (1.2), we share some common mathematical approaches.

For the system (KS), we refer recent paper [38] carrying Hölder regularity and uniqueness results (when $\alpha > 0$) relying on technical proofs originated from [9] and [18]. Compare to similar Hölder regularity results on [38], we play with a scalar equation (1.2) to obtain the same results under the weaker assumptions on B and ∇B that belongs to scaling invariant class. By following natural behaviour of a solution using a more geometrical approach (refer [32] and [33]), as a separate interest of its own, we provide a unified method of proof in the sense that the method has no limitation including $\alpha = 0$ (usually it is important to have $\alpha > 0$ in [38], [9], and [18] and showing stability when $\alpha \rightarrow 0$ is regarded as an another computational issue). Besides simplicity of computations in this manuscript, our method of proof carries potentials to provide significant common elements to the similar proofs for singular type of equations (when $-1 < \alpha \leq 0$) and even for generalized structured equations (refer Remark 3.4 for details).

Here we provide the definition of a weak solution of (1.2).

Definition 1.1. Let Ω be an open set in \mathbb{R}^d , $B \in L^2((0, T) \times \Omega)$, and $T > 0$.

$$n \in C_{\text{loc}}(0, T; L_{\text{loc}}^2(\Omega)), \quad n^{\frac{\alpha+2}{2}} \in L_{\text{loc}}^2(0, T; W_{\text{loc}}^{1,2}(\Omega))$$

is a local weak solution to (1.2) with $\alpha \geq 0$ if for every compact set $K \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$(1.5) \quad \int_K n\varphi dx \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_K \{-n\varphi_t + \nabla n^{1+\alpha} \nabla \varphi + Bn \nabla \varphi\} dx dt = 0$$

for all nonnegative testing functions

$$\varphi \in W_{\text{loc}}^{1,2}(0, T; L^2(K)) \cap L_{\text{loc}}^2(0, T; W_o^{1,2}(K)).$$

From the definition of weak solutions, we compute two types of energy estimates, provided in Propositions 4.1 and 4.2 which is called local and logarithmic energy estimates, respectively. Due to the difference of the nature of porous medium and p -Laplacian equations, we modify the method of proof in [32], for example, considering the a weak solution directly rather sub or super solutions, also cutting off a weak solution when u may stay near zero for DeGiorgi iteration. Moreover, another technical issue follows because the lower order term in (1.2) does not follow the structure of main term (not given in the form of $n^{1+\alpha}$ but n). By imposing conditions of B and ∇B in scaling invariant class, we can provide simpler proof compare to computation in [38]. Also conditions on ∇B does follow global estimates from (KS-PME).

Before we deliver the local Hölder continuity results, we make comments on intrinsic scaling due to the nonhomogeneity of the equation (1.2). More precisely, the local energy estimate derived from (1.2) appears in Proposition 4.1 is non-homogenous unless $\alpha = 0$. Roughly speaking, in an intrinsically (rescaled with the behaviour of a solution) scaled cylinder, a weak solution behaves like a solution to the heat equation. That is, more specifically, rescaling the time length

$$(1.6) \quad T_{\omega, \rho} = \theta \omega^{-\alpha} \rho^2$$

for some constant θ and ρ and

$$(1.7) \quad \omega := \operatorname{ess\,osc}_{\Omega_T} n = \mu_+ - \mu_- := \operatorname{ess\,sup}_{\Omega_T} n - \operatorname{ess\,inf}_{\Omega_T} n.$$

Since Ω_T is open, there are positive constants r and s such that $K_r^{x_0} \times (t_0 - s, t_0) \subset \Omega_T$. If we set

$$R = \frac{1}{4} \min \left\{ r, \frac{\omega^{\alpha/2} s^{1/2}}{\theta^{1/2}} \right\},$$

then we conclude that

$$Q_{\omega, 4R}^{x_0, t_0}(\theta) = K_{4R}^{x_0} \times (t_0 - \theta \omega^{-\alpha} R^2, t_0) \subset \Omega_T.$$

Then for any positive constants θ and ω , we can fit the cylinder $Q_{\omega, 4R}^{x_0, t_0}(\theta)$ in Ω_T by selecting R properly. Basically, we are going to work with the cylinder $Q_{\omega, 4R}^{x_0, t_0}(\theta)$ to find a proper subcylinder where a solution has less oscillation eventually leading to Hölder continuity.

Due to the intrinsic scaling (1.6), we define a time scale in terms of the function n and the set Ω on which n is defined. For any real number τ , we define

$$(1.8) \quad |\tau|_I = \omega^{\alpha/2} |\tau|^{1/2}.$$

With this time scale, we define the parabolic distance between two sets such \mathcal{K}_1 and \mathcal{K}_2 by

$$\operatorname{dist}_p(\mathcal{K}_1; \mathcal{K}_2) = \inf_p \max_{\substack{(x, t) \in \mathcal{K}_1 \\ (y, s) \in \mathcal{K}_2 \\ s \leq t}} \{|x - y|_\infty, |t - s|_I\}$$

with $|\cdot|_\infty$ (which is defined by $|x - y|_\infty = \max_{1 \leq i \leq d} |x^i - y^i|$).

Now we state the Hölder continuity of a bounded weak solution of (1.2).

Theorem 1.2. (*Hölder continuity of n*) *Let n be a nonnegative bounded weak solution of (1.2) under (1.3) with $\alpha \geq 0$ in Ω_T . Then n is locally continuous. Moreover, there exist positive constant $\beta \in (0, 1)$ and γ depending on data (that is, $d, \Omega_T, \Omega'_T, \alpha, \|B\|_{2\hat{q}_1, 2\hat{q}_2}, \|\nabla B\|_{\hat{q}_1, \hat{q}_2}$ for some $\hat{q}_1, \hat{q}_2 > 1$ satisfying (1.4)) such*

that, for any two distinct points (x_1, t_1) and (x_2, t_2) in any subset Ω'_T of Ω_T with $\text{dist}(\Omega'_T; \partial_p \Omega_T)$ positive, we have

$$(1.9) \quad |n(x_1, t_1) - n(x_2, t_2)| \leq \gamma \omega \left(\frac{|x_1 - x_2| + \omega^{\alpha/2} |t_1 - t_2|^{1/2}}{\text{dist}_p(\Omega'_T; \partial_p \Omega_T)} \right)^\beta.$$

The proof of this theorem is given in Section 3 considering two alternatives. Then the proofs of two alternatives are shown in Section 4 as combinations of DeGorgi iterations and the expansion of positivity along the time axis and the spatial axis.

Now we state results on the existence of global-intime weak solution of (KS-PME) and global Hölder continuity of the Cauchy problems of (KS-PME) as well. For the notational convenience, we denote

$$A := \{(q, \alpha) \mid \alpha > 2q - 2, q \geq 1\}, \quad B := \left\{ (q, \alpha) \mid \alpha > \frac{9q - 8}{6}, q \geq 1 \right\},$$

$$C := \left\{ (q, \alpha) \mid \alpha > \frac{10q - 9}{8}, q \geq 1 \right\}.$$

We introduce the notions of weak solutions and Hölder continuous solutions. We start with the definition of weak solutions.

Definition 1.3. (*Weak solutions*) Let $q \geq 1$ and $0 < T < \infty$. A triple (n, c, u) is said to be a weak solution of the system (1.1) if the followings are satisfied:

- (i) n and c are non-negative functions and u is a vector function defined in $\mathbb{R}^3 \times (0, T)$ such that

$$n(1 + |x| + |\log n|) \in L^\infty(0, T; L^1(\mathbb{R}^3)), \quad \nabla n^{\frac{1+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)),$$

$$c \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), \quad c \in L^\infty(\mathbb{R}^3 \times [0, T]),$$

$$u \in L^\infty(0, T; L^2(\mathbb{R}^3)), \quad \nabla u \in L^2(0, T; L^2(\mathbb{R}^3)),$$

- (ii) (n, c, u) satisfies the system (1.1) in the sense of distributions, namely,

$$\int_0^T \int_{\mathbb{R}^d} (n\varphi_t - \nabla n^{1+\alpha} \cdot \nabla \varphi + nu \cdot \nabla \varphi + n^q \chi(c) \nabla c \cdot \nabla \varphi) dx dt = - \int_{\mathbb{R}^d} n_0 \varphi(\cdot, 0) dx,$$

$$\int_0^T \int_{\mathbb{R}^d} (c\varphi_t - \nabla c \cdot \nabla \varphi + cu \cdot \nabla \varphi + n\kappa(c)\varphi) dx dt = - \int_{\mathbb{R}^d} c_0 \varphi(\cdot, 0) dx,$$

$$\int_0^T \int_{\mathbb{R}^d} (u \cdot \psi_t - \nabla u \cdot \nabla \psi + (\tau(u \cdot \nabla \psi)) u - n \nabla \phi \cdot \psi) dx dt = - \int_{\mathbb{R}^d} u_0 \cdot \psi(\cdot, 0) dx$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3 \times [0, T])$ and $\psi \in C_0^\infty(\mathbb{R}^3 \times [0, T], \mathbb{R}^3)$ with $\nabla \cdot \psi = 0$.

Next we define Hölder continuous solutions. For convenience, we denote $Q_T := (0, T) \times \mathbb{R}^3$.

Definition 1.4. (*Hölder continuous solutions*) Let $q \geq 1$, $(q, \alpha) \in (A \cap B)$ and $0 < T < \infty$. A triple (n, c, u) is said to be a Hölder continuous solutions of the system (1.1) if (n, c, u) is a weak solution in Definition 1.3 and furthermore satisfies the following: there exists $\beta > 0$ such that

$$(1.10) \quad n, \partial_t c, \partial_t u, \nabla^2 c, \nabla^2 u \in C^\beta(Q_T).$$

Before stating our result precisely, we first recall some essential conditions for χ and κ . To preserve the non-negativity of the density of bacteria $n(x, t)$ and the oxygen $c(x, t)$ for $0 < t < T$, it is necessary to assume that $\kappa(0) = 0$. The condition $\kappa(\cdot) \geq 0$ is also essential since the bacteria consume the oxygen. Thus, the following hypotheses are compulsory: $\kappa(\cdot) \geq 0$ and $\kappa(0) = 0$. Furthermore, we suppose that $\chi' \in L_{\text{loc}}^\infty$. Summing up, throughout this thesis, we assume that

$$(P_1) \quad \chi' \in L_{\text{loc}}^\infty, \quad \kappa \in L_{\text{loc}}^\infty, \quad \kappa(\cdot) \geq 0 \quad \text{and} \quad \kappa(0) = 0.$$

To obtain more extended range of α , we sometimes make further assumptions on κ , which are given by

$$(P_2) \quad \kappa' \in L_{\text{loc}}^\infty \quad \text{with} \quad \kappa'(\cdot) \geq \kappa_0 \quad \text{for some constant } \kappa_0 > 0.$$

We now present two different types of assumptions on χ, κ together with the range of α and q . The first one is reserved for weak solutions.

Assumption 1.5. χ, κ and α satisfy (P_1) and one of the following holds:

- (i) $(q, \alpha) \in B$.
- (ii) $(q, \alpha) \in A \cup B$ and κ satisfies (P_2) .

Next assumption is prepared for Hölder continuous solutions.

Assumption 1.6. χ, κ and α satisfy (P_1) and one of the following holds:

- (i) $(q, \alpha) \in A \cap B$.
- (ii) $(q, \alpha) \in (A \cup B) \cap C$ and κ satisfies (P_2) .

We recall some known results related to our concerns. Firstly, we compare the system (KS-PME) to the classical Keller-Segel model (KS) of porous medium type, which is given as

$$(1.11) \quad (\text{KS}) \quad \begin{cases} \partial_t n = \Delta n^{1+\alpha} - \nabla \cdot (\chi n \nabla c), \\ \tau \partial_t c = \Delta c - c + n, \end{cases}$$

where χ is a positive constant, $q = 1$ and $\tau = 0$ or 1 (for example, [36, 37]). We remark that the equation of c in (1.11) is modeled by the chemical substance, which is produced by biological organism, but in our case the equation (1.1)₂ indicates the dynamics of oxygen, which is consumed by a certain type of bacteria. That's the reason opposite sign of the right side of each equation appears, which causes main difference regarding global existence or blow-up for the value on α . In case that (1.11), the equation of c , is of elliptic type, i.e. $\tau = 0$, existence of bounded weak solutions was shown in [52] globally in time, provided that $q \geq 1$ and $\alpha > q - \frac{2}{d}$. If $0 < \alpha \leq q - \frac{2}{d}$, blow-up may occur in a finite time. Later, in [34], the result of [52] was extended to the case that the equation of c is of parabolic type, i.e. $\tau > 0$.

For the chemotaxis fluid system (1.1) with $q = 1$ in two dimensions, it was known that bounded weak solutions exist globally in time under some assumptions on κ and χ for sufficiently regular data. We remark that results in dimension two are even valid in replacement with the Navier-Stokes equations for fluid equations (refer to [12] and [53])

In three dimensions, it was shown in [43] that if $\alpha = \frac{1}{3}$, then the chemotaxis-Stokes system (1.1) with $q = 1$ has global-in-time bounded weak solutions. For the special case that $\chi = 1$ and $\kappa(x) = x$, existence of bounded weak solutions was proved in [54] for (KS-PME) with $q = 1$, provided that $\alpha > 1/7$. In [12], for (1.1) with $q = 1$, it was proved that global-in-time existence of weak solutions and

TABLE 1. Relations between parameters and conditions

	weak solutions	Hölder continuous solutions
(P_1)	$\alpha > \frac{9q-8}{6}$ (Theorem 1.7)	$\alpha > \max \left\{ 2q - 2, \frac{9q-8}{6} \right\}$ (Theorem 1.9)
(P_1) and (P_2)	$\alpha > \min \left\{ 2q - 2, \frac{9q-8}{6} \right\}$ (Theorem 1.8)	$\alpha > \max \left\{ \min \left\{ 2q - 2, \frac{9q-8}{6} \right\}, \frac{10q-9}{8} \right\}$ (Theorem 1.10)

bounded weak solutions under the same conditions as (P_1) and (P_2) , if $\alpha > 1/6$ and $\alpha > 1/4$, respectively. The range of α was improved in [11]. More precisely, if $\alpha > \frac{1}{6}$, bounded weak solutions exist under only the condition (P_1) . Furthermore, it was also proved that if χ or κ satisfy (P_1) and (P_2) , and if $\alpha > \frac{1}{8}$, then there exists bounded weak solutions for the system (1.1) with $q = 1$.

As mentioned earlier, our main goal is to study more general Keller-segel-fluid system (1.1) with $q \geq 1$ and obtain global existence of weak and Hölder continuous solutions for extended range of α and q . Our results are summarized in the Table 1. We remark that in case that $q = 1$, our results recover those of [11].

Now we are ready to state our main results of the system (1.1), and the first one is about existence of weak solutions, which reads as follows:

Theorem 1.7. (*Weak solutions*) Let α belong to B i.e., $\alpha > \frac{9q-8}{6}$ and initial data (n_0, c_0, u_0) satisfy

$$(1.12) \quad n_0(1 + |x| + |\log n_0|) \in L^1(\mathbb{R}^3), \quad c_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \text{ and } u_0 \in L^2(\mathbb{R}^3).$$

Suppose that χ, κ satisfy the hypothesis (P_1) . Then, there exists a weak solution (n, c, u) for the system (1.1). Furthermore, for any p with $1 \leq p \leq \alpha - q + 2$

$$n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)),$$

and the following inequality is satisfied:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n(|\log n| + \langle x \rangle) + \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\ & + C \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + \|\Delta c\|_2^2 + \|\nabla u\|_2^2 \right) \\ & \leq C \left(T, \|c_0\|_{L^\infty \cap H^1}, \|n_0(1 + |x| + |\log n_0|)\|_1, \|n_0\|_{\alpha-q+2}, \|u_0\|_2 \right), \end{aligned}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

If the condition (P_2) is additionally assumed, the range of α is a bit expanded, compared to that of Theorem 1.7. More precisely, we have the following:

Theorem 1.8. (*Weak solutions*) Let α belong to $A \cup B$ i.e., $\alpha > \min \left\{ 2q - 2, \frac{9q-8}{6} \right\}$. Suppose that χ, κ satisfy the hypothesis (P_1) , (P_2) and initial data (n_0, c_0, u_0) satisfies

$$(1.13) \quad n_0(1 + |x| + |\log n_0|) \in L^1(\mathbb{R}^3), \quad c_0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \text{ and } u_0 \in H^1(\mathbb{R}^3).$$

Then, there exists a weak solution (n, c, u) for the system (KS-PME). Furthermore, for any p with $1 \leq p \leq \alpha - 2q + 3$

$$n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad n^{\frac{1}{2}} \nabla c \in L^2(0, T; L^2(\mathbb{R}^3)), \quad \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3)),$$

and the following inequality is satisfied:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n (|\log n| + \langle x \rangle) + \int_{\mathbb{R}^3} n^{\alpha-2q+3} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\ & + C \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^2 + \|\Delta c\|_2^2 + \|\nabla u\|_2^2 \right) \\ & \leq C \left(T, \|\nabla c_0\|_2, \|n_0 \log n_0\|_1, \|n_0\|_{\alpha-2q+3}, \|u_0\|_2 \right), \end{aligned}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

Next, if α is greater than a certain value depending on q , we prove existence of Hölder continuous solutions for (KS-PME) under the condition (P_1) . To be more precise, the result reads as follows:

Theorem 1.9. (*Hölder continuous solutions*) Let α belongs to $A \cap B$ i.e., $\alpha > \max\{2q-2, \frac{9q-8}{6}\}$. Suppose that χ, κ satisfy the hypothesis (P_1) and initial data (n_0, c_0, u_0) satisfies (1.12) as well as

$$(1.14) \quad n_0 \in L^\infty(\mathbb{R}^3), \quad c_0 \in W^{1,m}(\mathbb{R}^3), \quad u_0 \in W^{1,m}(\mathbb{R}^3), \quad \text{for any } m < \infty.$$

Then, there exists a Hölder continuous solution (n, c, u) for the system (KS-PME).

Furthermore, we assume the condition (P_2) and we then see that the restriction of α is relaxed for the existence of Hölder continuous solutions.

Theorem 1.10. (*Hölder continuous solutions*) Let α belongs to $(A \cup B) \cap C$ i.e., $\alpha > \max\{\min\{2q-2, \frac{9q-8}{6}\}, \frac{10q-9}{8}\}$. Suppose that χ, κ satisfy the hypothesis $(P_1), (P_2)$ and initial data (n_0, c_0, u_0) satisfies (1.13) as well as

$$(1.15) \quad n_0 \in L^\infty(\mathbb{R}^3), \quad c_0 \in W^{1,m}(\mathbb{R}^3), \quad u_0 \in W^{1,m}(\mathbb{R}^3), \quad \text{for any } m < \infty.$$

Then, there exists a Hölder continuous solution (n, c, u) for the system (KS-PME).

Remark 1.11. There are some known results regarding uniqueness of Hölder continuous solutions for Keller-Segel system of porous medium type (see e.g. [46] and [38]). As for us, uniqueness of solutions in Theorem 1.9 and Theorem 1.10 doesn't seem to be obvious, in particular, due to presence of the fluid velocity field. Therefore, we leave it as an open question.

This paper is organized as follows: In section 2, we introduce some notations and review known results. Section 3 is devoted for the proof of Theorem 1.2 with the crucial aid of two alternatives, whose are clarified in section 4. In section 5, we present the proofs of existence for weak solutions of (KS-PME) in dimension three. We also provide the proofs of Theorem 1.9 and Theorem 1.10 in section 6. In appendix, proofs of Propositions 4.1 and 4.2 are given.

2. PRELIMINARIES

2.1. Notations and useful inequalities. In this subsection, We introduce the notations throughout this paper and recall some useful inequalities for our purpose. Let Ω be an open domain in \mathbb{R}^d , $d \geq 1$ and I a finite interval.

$$L^p(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ is Lebesgue measurable, } \|f\|_{L^p(\Omega)} < \infty\},$$

where

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f|^p dx \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty).$$

We will write $\|f\|_{L^p(\Omega)} := \|f\|_p$, unless there is any confusion to be expected. For $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ denotes the usual Sobolev space, i.e.,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega), 0 \leq |\alpha| \leq k\}.$$

We also write the mixed norm of f in spatial and temporal variables as

$$\|f\|_{L_{x,t}^{p,q}(\Omega \times I)} = \|f\|_{L_t^q(I; L_x^p(\Omega))} = \left\| \|f\|_{L_x^p(\Omega)} \right\|_{L_t^q(I)}.$$

Let m and p be positive constants greater than 1 and consider the Banach spaces

$$V^{m,p}(\Omega_T) := L^{\infty}(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$$

and

$$V_0^{m,p}(\Omega_T) := L^{\infty}(0, T; L^m(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)),$$

both equipped with the norm $v \in V^{m,p}(\Omega_T)$,

$$\|v\|_{V^{m,p}(\Omega_T)} := \operatorname{ess\,sup}_{0 < t < T} \|v(\cdot, t)\|_{m, \Omega} + \|\nabla v\|_{p, \Omega_T}.$$

When $m = p$, we set $V^{p,p}(\Omega_T) = V^p(\Omega_T)$. Note that both spaces are embedded in $L^q(\Omega_T)$ for some $q > p$. We denote by $C = C(\alpha, \beta, \dots)$ a constant depending on the prescribed quantities α, β, \dots , which may change from line to line.

Now we introduce basic embedding inequalities and auxiliary lemmas for fast geometric convergence. (Refer Chapter I in [18])

Theorem 2.1. (*Gagliardo-Nirenberg multiplicative embedding inequality*) *Let $v \in W_0^{1,p}(\Omega)$, $p \geq 1$. For every fixed number $s \geq 1$ there exists a constant C depending only upon d, p and s such that*

$$\|v\|_{q, \Omega} \leq C \|\nabla v\|_{p, \Omega}^{\alpha} \|v\|_{s, \Omega}^{1-\alpha},$$

where $\alpha \in [0, 1]$, $p, q \geq 1$, are linked by

$$\alpha = \left(\frac{1}{s} - \frac{1}{q} \right) \left(\frac{1}{d} - \frac{1}{p} + \frac{1}{s} \right)^{-1},$$

and their admissible range is

$$\begin{cases} q \in [s, \infty], \alpha \in [0, \frac{p}{p+s(p-1)}], & \text{if } d = 1, \\ q \in [s, \frac{dp}{d-p}], \alpha \in [0, 1], & \text{if } 1 \leq p < d, \ s \leq \frac{dp}{d-p}, \\ q \in [\frac{dp}{d-p}, s], \alpha \in [0, 1], & \text{if } 1 \leq p < d, \ s \geq \frac{dp}{d-p}, \\ q \in [s, \infty), \alpha \in [0, \frac{dp}{dp+s(p-d)}], & \text{if } 1 < d \leq p. \end{cases}$$

Theorem 2.2. (*Sobolev embedding theorem*) *There exists a constant C depending only upon d, p, m such that for every $v \in L^{\infty}(0, T; L^m(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$,*

$$\iint_{\Omega_T} |v(x, t)|^q dx dt \leq C^q \left(\operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} |v(x, t)|^m dx \right)^{p/d} \left(\iint_{\Omega_T} |\nabla v(x, t)|^p dx dt \right)$$

where $q = \frac{p(d+m)}{d}$. Moreover,

$$\|v\|_{q, \Omega_T} \leq C \|v\|_{V^{m,p}(\Omega_T)} = C \left(\operatorname{ess\,sup}_{0 < t < T} \|v(\cdot, t)\|_{m, \Omega} + \|\nabla v\|_{p, \Omega_T} \right).$$

When $p = m$, we apply Hölder's inequality to obtain the following corollary.

Corollary 2.3. *Let $p > 1$. There exists a constant C depending only upon d and p , such that for every $v \in V_0^p(\Omega_T)$,*

$$\|v\|_{p, \Omega_T}^p \leq C |\{|v| > 0\}|^{\frac{p}{d+p}} \|v\|_{V^p(\Omega_T)}^p.$$

Proposition 2.4. *There exists a constant C depending only upon d and p such that for every $v \in V_0^p(\Omega_T)$,*

$$\|v\|_{q_1, q_2; \Omega_T} \leq C \|v\|_{V^p(\Omega_T)}$$

where the numbers $q_1, q_2 \geq 1$ are linked by

$$\frac{1}{q_2} + \frac{d}{pq_1} = \frac{d}{p^2},$$

and their admissible range is

$$\begin{cases} q_1 \in (p, \infty), & q_2 \in (p^2, \infty); & \text{for } d = 1, \\ q_1 \in (p, \frac{dp}{d-p}), & q_2 \in (p, \infty); & \text{for } 1 < p < d, \\ q_1 \in (p, \infty), & q_2 \in (\frac{p^2}{d}, \infty); & \text{for } 1 < d \leq p. \end{cases}$$

Proof. Let $v \in V_0^p(\Omega_T)$ and let $r \geq 1$ to be chosen. From Theorem 2.1 with $s = p$ follows that

$$\begin{aligned} & \left(\int_0^T \|v(\cdot, \tau)\|_{q_1, \Omega}^{q_2} d\tau \right)^{1/r} \\ & \leq C \left(\int_0^T \|\nabla v(\cdot, \tau)\|_p^{\alpha q_2} d\tau \right)^{1/r} \operatorname{ess\,sup}_{0 \leq \tau \leq T} \|v(\cdot, \tau)\|_{p, \Omega_T}^{1-\alpha}. \end{aligned}$$

Choose α such that $\alpha q_2 = p$. □

We state a lemma concerning the geometric convergence of sequences of numbers.

Lemma 2.5. *Let $\{Y_n\}$ and $\{Z_n\}$, $n = 0, 1, 2, \dots$, be sequences of positive numbers, satisfying the recursive inequalities*

$$\begin{cases} Y_{n+1} \leq C b^n (Y_n^{1+\alpha} + Z_n^{1+\kappa} Y_n^\alpha) \\ Z_{n+1} \leq C b^n (Y_n + Z_n^{1+\kappa}) \end{cases}$$

where $C, b > 1$ and $\kappa, \alpha > 0$ are given numbers. If

$$Y_0 + Z_0^{1+\kappa} \leq (2C)^{-\frac{1+\kappa}{\sigma}} b^{-\frac{1+\kappa}{\sigma^2}}, \quad \text{where } \sigma = \min\{\kappa, \alpha\},$$

then $\{Y_n\}$ and $\{Z_n\}$ tend to zero as $n \rightarrow \infty$.

The following lemma is introduced in [19]; it states that if the set where v is bounded away from zero occupies a sizable portion of K_ρ , then the set where v is positive cluster about at least one point of K_ρ . Here we name the inequality as the isoperimetric inequality.

Lemma 2.6. *(Isoperimetric inequality) Let $v \in W^{1,1}(K_\rho^{x_0}) \cap C(K_\rho^{x_0})$ for some $\rho > 0$ and some $x_0 \in \mathbb{R}^d$ and let k and l be any pair of real numbers such that*

$k < l$. Then there exists a constant γ depending upon N, p and independent of k, l, v, x_0, ρ , such that

$$(l - k)|K_\rho^{x_0} \cap \{v > l\}| \leq \gamma \frac{\rho^{d+1}}{|K_\rho^{x_0} \cap \{v \leq k\}|} \int_{K_\rho^{x_0} \cap \{k < v < l\}} |Dv| dx.$$

We consider the following heat equation:

$$(2.1) \quad v_t - \Delta v = f, \quad \text{in } Q_T := \mathbb{R}^3 \times (0, T),$$

with initial data $v(x, 0) = v_0(x)$. Next, we recall maximal estimates of the heat equation in terms of mixed norms.

Lemma 2.7. *Let $1 < l, m < \infty$. Suppose that $f \in L_{x,t}^{l,m}(Q_T)$ and $v_0 \in W^{2,l}(\mathbb{R}^3)$. If v is the solution of the heat equation (2.1), then the following estimate is satisfied:*

$$(2.2) \quad \|v_t\|_{L_{x,t}^{l,m}(Q_T)} + \|\nabla^2 v\|_{L_{x,t}^{l,m}(Q_T)} \leq C \left(\|f\|_{L_{x,t}^{l,m}(Q_T)} + \|v_0\|_{W^{2,l}(\mathbb{R}^3)} \right).$$

We also recall the following Stokes system, which is the linearized Stokes equations:

$$(2.3) \quad v_t - \Delta v + \nabla p = f, \quad \operatorname{div} v = 0 \quad \text{in } Q_T := \mathbb{R}^3 \times (0, T),$$

with initial data $v(x, 0) = v_0(x)$. The maximal estimates of the Stokes system is given as follows.

Lemma 2.8. *Let $1 < l, m < \infty$. Suppose that $f \in L_{x,t}^{l,m}(Q_T)$ and $v_0 \in W^{2,l}(\mathbb{R}^3)$. If v is the solution of the Stokes system (2.3), then the following estimate is satisfied:*

$$(2.4) \quad \|v_t\|_{L_{x,t}^{l,m}(Q_T)} + \|\nabla^2 v\|_{L_{x,t}^{l,m}(Q_T)} + \|\nabla p\|_{L_{x,t}^{l,m}(Q_T)} \leq C \left(\|f\|_{L_{x,t}^{l,m}(Q_T)} + \|v_0\|_{W^{2,l}(\mathbb{R}^3)} \right).$$

3. PROOF OF THEOREM 1.2

For notational convention, we take ν_0 to be the constant from Proposition 4.3 (DeGiorgi type of iteration) corresponding to $\theta = 1$ and, with ω and R given positive constants, we set

$$(3.1) \quad \Delta = \left(\frac{\omega}{2} \right)^{-\alpha} (2R)^2.$$

Our first alternative is that, if a bounded weak solution n stays close to its maximum on most of one suitable small subcylinder, then n is away from its minimum on a suitable subcylinder centered at $(0, 0)$, the target point.

Moreover, for given constants k (in analysis it denotes the level of solution) and ρ (usually it means the spacial radius), we assume that

$$(3.2) \quad k^{-\alpha - \frac{2\alpha(1+\kappa)}{q_2}} \rho^{d\kappa} < 1.$$

If (3.2) fails, we have $k \leq \rho^\epsilon$ for some $\epsilon > 0$ which directly implies the Hölder continuity of a solution.

Lemma 3.1. *(The first alternative) Let $\theta_0 > 1$ be a given constant, ν_0 be a constant in Proposition 4.3 (when $\theta = 1$) and Δ be in (3.1). Suppose n is a nonnegative bounded weak solution of (1.2) in*

$$(3.3) \quad Q = K_{2R} \times (-\theta_0 \Delta, 0)$$

with $\alpha \geq 0$. If there is a constant $T_0 \in [-\theta_0\Delta, -\Delta]$ such that

$$(3.4) \quad \left| K_{2R} \times (T_0, T_0 + \Delta) \cap \{n < \mu_- + \frac{\omega}{2}\} \right| \leq \nu_0 |K_{2R}| \Delta,$$

then there is a constant $\delta_1 \in (0, 1)$ determined only by θ_0 and data such that

$$\operatorname{ess\,inf}_{Q'} n(x, t) \geq \mu_- + \delta_1 \omega$$

where

$$(3.5) \quad Q' = K_{R/2} \times [-\omega^{-\alpha} \left(\frac{R}{2}\right)^2, 0].$$

When the assumption (3.4) fails, which means that n stays somewhat close to its maximum on a suitable fraction of all suitable small subcylinders, then eventually n is away from its supremum on a suitable subcylinder centered at $(0, 0)$.

Lemma 3.2. *(The second alternative) There are constants $\theta_0 > 1$ and $\nu_0 \in (0, 1)$ determined only by data such that, if n is a bounded weak solution of (1.2) in Q (given by (3.3)) with $\alpha \geq 0$ and*

$$(3.6) \quad \left| K_{2R} \times (T_0, T_0 + \Delta) \cap \{n > \mu_+ - \frac{\omega}{2}\} \right| \leq (1 - \nu_0) |K_{2R}| \Delta,$$

for all $T_0 \in [-\theta_0\Delta, -\Delta]$, then there is a constant $\delta_2 \in (0, 1)$ determined only by data, such that

$$\operatorname{ess\,sup}_{Q'} n \leq \mu_+ - \delta_2 \omega$$

where Q' is given in (3.5).

Also we prove this lemma in Section 4. From two alternatives, we infer a decay estimate for the oscillation of a bounded nonnegative weak solution of (1.2).

Lemma 3.3. *(Main Lemma) Let $\alpha, \rho, \mu_+, \omega$ be given constants with $\alpha > 0$. Suppose also that n is a nonnegative bounded weak solution of (1.2) in $Q_{\omega, \rho} = K_\rho \times [-\omega^{-\alpha} \rho^2, 0]$ with $\operatorname{ess\,osc}_{Q_{\omega, \rho}} n \leq \omega$. Then there are positive constants η and λ , both less than one and determined only by data such that*

$$\operatorname{ess\,osc}_{Q_{\eta\omega, \lambda\rho}} n \leq \eta\omega$$

where

$$Q_{\eta\omega, \lambda\rho} = K_{\lambda\rho} \times [-(\eta\omega)^{-\alpha} (\lambda\rho)^2, 0].$$

Proof. Let us call Q and Q' where Lemma 3.1 and Lemma 3.2 hold. Then, here, our goal is to choose proper R and λ such that

$$Q_{\eta\omega, \lambda\rho} \subseteq Q' \subset Q \subseteq Q_{\omega, \rho}.$$

Then the proper relationship of two essential oscillation is following rather directly from two alternatives, Lemma 3.1 and Lemma 3.2.

For $\theta_0 > 1$ given in both Lemma 3.1 and Lemma 3.2, we choose

$$R = \theta_0^{-1/2} 2^{-\alpha/2-1} \rho < \rho/2.$$

Then $Q \subseteq Q_{\omega, \rho}$. For $\eta = \max\{1 - \delta_1, 1 - \delta_2\} \in (0, 1)$ where δ_1 and δ_2 are from Lemma 3.1 and Lemma 3.2, we choose

$$(3.7) \quad \lambda = \eta^{\alpha/2} \theta_0^{-1/2} 2^{-\alpha/2-2}$$

so that $Q_{\eta\omega, \lambda\rho} \subseteq Q'$.

If there is a $T_0 \in (-\theta_0\Delta, -\Delta)$ such that

$$(3.8) \quad \left| K_{2R} \times (T_0, T_0 + \Delta) \cap \{n > \mu_+ - \frac{\omega}{2}\} \right| \leq \nu_0 |K_{2R}| \Delta,$$

then by Lemma 3.1 applied to n implies that

$$\operatorname{ess\,sup}_{Q'} n \leq \mu_+ - \delta_1 \omega.$$

Hence, by Lemma 3.1 and $Q_{\eta\omega, \lambda\rho} \subseteq Q'$, it follows that

$$\operatorname{ess\,osc}_{Q_{\eta\omega, \lambda\rho}} n \leq \operatorname{ess\,osc}_{Q'} n \leq \eta\omega.$$

When (3.8) fails, then it holds

$$\left| K_{2R} \times (T_0, T_0 + \Delta) \cap \{n < \mu_- + \frac{\omega}{2}\} \right| \leq (1 - \nu_0) |K_{2R}| \Delta.$$

By Lemma 3.2, we have

$$\operatorname{ess\,inf}_{Q'} n \geq \mu_- + \delta_2 \omega$$

which implies that

$$\operatorname{ess\,osc}_{Q_{\eta\omega, \lambda\rho}} n \leq \eta\omega.$$

This completes the proof. \square

Now we provide the proof of Theorem 1.2.

Proof of Theorem 1.2: If $\omega = 0$, then this result is true for any choice of γ and β , so we assume that $\omega > 0$ and set $\omega_0 = \omega$. We also set

$$\rho_0 = \operatorname{dist}_p(\{(x_1, t_1)\}, \partial_p \Omega_T).$$

We define

$$\omega_i = \eta^i \omega_0, \quad \rho_i = \lambda^i \rho_0,$$

where λ and η are the constants from Lemma 3.3. Also define a sequence of cylinders $\{Q_n\}$ by

$$Q_i = K_{\rho_i}^{x_1} \times [t_1 - \omega_i^{-\alpha} \rho_i^2, t_1].$$

It is easy to check that $Q_0 \subset \Omega$ and that $Q_{i+1} \subset Q_i$ for any i . Combining with Lemma 3.3 with an induction argument, we find that $\operatorname{ess\,osc}_{Q_i} n \leq \omega_i$ for any i .

For $(x_2, t_2) \in Q_0$ with $x_1 \neq x_2$ and $t_1 \neq t_2$, then there are nonnegative integers k and l such that

$$(3.9) \quad \rho_{k+1} < |x_1 - x_2| \leq \rho_k,$$

and

$$(3.10) \quad \omega_{l+1}^{-\alpha} \rho_{l+1}^2 < |t_1 - t_2| \leq \omega_l^{-\alpha} \rho_l^2.$$

As a result, we obtain that

$$|n(x_1, t_1) - n(x_2, t_2)| \leq \max\{\omega_k, \omega_l\}.$$

From (3.9), we derive, for $\beta_1 = \log_\eta \lambda$,

$$\frac{|x_1 - x_2|}{\rho_0} > \lambda^{k+1} = (\eta^{\beta_1})^{k+1}$$

which implies

$$\omega_k = \eta^k \omega_0 < \eta \omega_0 \left(\frac{|x_1 - x_2|}{\rho_0} \right)^{\beta_1}.$$

On the other hand, the inequality (3.10) implies that

$$|t_1 - t_2|_I > \eta^{-\alpha(l+1)/2} \rho_{l+1} = \left(\eta^{-\alpha/2} \lambda \right)^{l+1} \rho_0.$$

Because of the choice of λ from (3.7), let us denote that

$$\tilde{\lambda} = \eta^{-\alpha/2} \lambda < 1.$$

Then we have

$$\omega_l = \tilde{\lambda}^{\beta_2 l} \omega_0 \leq \left(\frac{|t_1 - t_2|_I}{\tilde{\lambda} \rho_0} \right)^{\beta_2} \omega_0$$

for $\beta_2 = \log_{\tilde{\lambda}} \eta$.

Therefore, for some $\gamma > 0$,

$$|n(x_1, t_1) - n(x_2, t_2)| \leq \gamma \omega \left[\left(\frac{|x_1 - x_2|}{\rho_0} \right)^{\beta_1} + \left(\frac{|t_1 - t_2|_I}{\rho_0} \right)^{\beta_2} \right].$$

Then this implies (1.9) with $\beta = \min\{\beta_1, \beta_2\}$ and the definition of $|\cdot|_I$ (1.8) because $\rho_0 \geq \text{dist}_p(\Omega'; \partial_p \Omega)$.

If $x_1 = x_2$ or $t_1 = t_2$, then a similar and simpler arguments yields the same result.

Remark 3.4. Here we make comments that our method of analysis to show the local Hölder continuity of (1.2) can easily modified to explain the same regularity for a generalized structured equation. Now consider porous medium equation in the form of

$$(3.11) \quad n_t = \Delta \Phi(n) + \nabla \cdot (B(x, t)n)$$

where $\Phi \in C^1[0, \infty)$. Let $\Phi'(s) = \phi(s)$ where ϕ is a nonnegative increasing function with $\phi(0) = 0$ and we assume that there are two constants α_0 and α_1 satisfying $0 \leq \alpha_0 \leq \alpha_1 < \infty$ such that

$$(1 + \alpha_0)\Phi(s) \leq s\phi(s) \leq (1 + \alpha_1)\Phi(s)$$

for all $s > 0$. The two inequalities are essentially the ∇_2 and Δ_2 conditions in Orlicz space theory. If $\alpha_0 = \alpha_1$, then (3.11) is exactly (1.2).

The local and logarithmic energy estimates are obtained for (3.11) by replcing n^α to $\phi(n)$ in the estimates from Propositions 4.1 and 4.2. Hence corresponding intrinsic scaling for (3.11) follows immediately as $T_{\omega, \rho} = \theta \rho^2 / \phi(\omega)$ (cf. (1.6)). Moreover, we assume

$$\phi(k)^{-1 - \frac{2(1+\kappa)}{q_2}} \rho^{d\kappa} < 1$$

instead of (3.2). Then the same proofs in Sections 3 and 4 hold for (3.11) as well.

4. PROOFS OF THE TWO ALTERNATIVES

In this section, we deliver the proofs of two alternatives, Lemmas 3.1 and 3.2. The proofs in Section 4.4 are basically composed with two parts, DeGiorgi type iteration and the expansion of positivities along the time and space variables.

4.1. Local energy estimates. In this section, we provide two types of local energy estimates that are key to prove the modulus of continuity of n . We make two remarks. First, to carry calculations directly with weak solutions rather than sub(super-)solutions, we take advantage that both $n - \mu_-$ and $\mu_+ - n$ are nonnegative, which leads to the positiveness of level k . Second, because the lower order term does not appear in the form of $n^{1+\alpha}$ in (1.2), we assume proper conditions on B and ∇B (cf. Chapter 3.6 of [21]).

For given positive constants k and ρ , we denote the set

$$(4.1) \quad A_{k,\rho}^\pm(\tau) = \{x \in K_\rho : (n(x, \tau) - \mu_\pm \pm k)_\pm > 0\},$$

that indicates a level set (either $n < \mu_- + k$ or $n > \mu_+ - k$) at a fixed time τ .

Proposition 4.1. *Suppose that ζ is a cutoff function on the parabolic cylinder $Q_\rho = K_\rho \times [t_0, t_1]$, vanishing on the parabolic boundary of Q_ρ with $0 \leq \zeta \leq 1$. For a nonnegative bounded weak solution n of (1.2) under (1.3), it follows, for any $k > 0$ and for some positive constants*

$$(4.2) \quad \begin{aligned} & \sup_{t_0 \leq t \leq t_1} \int_{K_\rho \times \{t\}} (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx + (1 + \alpha) \iint_{Q_\rho} n^\alpha |\nabla(n - \mu_\pm \pm k)_\pm|^2 \zeta^2 dx dt \\ & \leq \int_{K_\rho \times \{t_0\}} (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx + 2 \iint_{Q_\rho} (n - \mu_\pm \pm k)_\pm^2 \zeta \zeta_t dx dt \\ & \quad + 16(1 + \alpha) \iint_{Q_\rho} (n^\alpha + k^\alpha) (n - \mu_\pm \pm k)_\pm^2 |\nabla \zeta|^2 dx dt \\ & \quad + (k^{2-\alpha} \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 + k^2 \|\nabla B\|_{\hat{q}_1, \hat{q}_2}) \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^\pm(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}. \end{aligned}$$

We deliver the proof in Section 7.1.

Now we provide a logarithmic energy estimate that is crucial to capture the expansion of positivity along the time axis, Proposition 4.6.

For given positive constants k and δ , let us define

$$(4.3) \quad \Psi_\pm(n) = \ln^+ \left[\frac{k}{(1 + \delta)k - (n - \mu_\pm \pm k)_\pm} \right].$$

By assuming proper conditions on B and ∇B given in (1.3), we succeed to calculate logarithmic estimates (refer Section B.7 on [21]).

Proposition 4.2. *For n a nonnegative weak solution of (1.2) with $\alpha \geq 0$ under (1.3), suppose that ζ is a cutoff function in $K_\rho \times [t_0, t_1]$ which is vanishing on the lateral boundary of $K_\rho \times [t_0, t_1]$ (independent of the time variable) with $0 \leq \zeta \leq 1$.*

For given positive constants k and δ , it follows that

$$\begin{aligned}
& \int_{K_\rho \times \{t_1\}} \Psi_\pm^2(n) \zeta^2 dx + 2(1+\alpha) \int_{t_0}^{t_1} \int_{K_\rho} n^\alpha |\nabla n|^2 \Psi_\pm(n) (\Psi'_\pm(n))^2 \zeta^2 dx dt \\
& \leq \int_{K_\rho \times \{t_0\}} \Psi_\pm^2(n) \zeta^2 dx + 16(1+\alpha) \int_{t_0}^{t_1} \int_{K_\rho} (n^\alpha + k^\alpha) \Psi_\pm^2(n) |\nabla \zeta|^2 dx dt \\
(4.4) \quad & + 2k^{-\alpha} \left(\ln \frac{1}{\delta} \right)^2 \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^\pm(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}} \\
& + \left(\left(\ln \frac{1}{\delta} \right)^2 + \frac{\mu_+ \ln \frac{1}{\delta}}{\delta k} \right) \|\nabla B\|_{q_1, q_2} \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^\pm(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}.
\end{aligned}$$

We deliver the proof in Section 7.1.

4.2. DeGiorgi type of estimates. In this section, we modify DeGiorgi iteration for the porous medium equations given in (1.2) with $\alpha \geq 0$ that starts from the local energy estimates given in Proposition 4.1. (refer Lemma 7.1 of [21])

Proposition 4.3. (*DeGiorgi iteration*) *Let n be a bounded nonnegative weak solution of (1.2) under (1.3) with $\alpha \geq 0$. For given positive constants ρ , k , and θ satisfying (3.2), let*

$$Q_\rho = Q(\rho, \theta k^{-\alpha} \rho^2) = K_\rho \times [-\theta k^{-\alpha} \rho^2, 0].$$

(i) *There exists $\nu_0 = \nu_0(\theta, \text{data}) \in (0, 1)$ such that, if*

$$(4.5) \quad |Q_\rho \cap \{n > \mu_+ - k\}| \leq \nu_0 |Q_\rho|,$$

then it holds

$$(4.6) \quad \operatorname{ess\,sup}_{Q_{\rho/2}} n(x, t) \leq \mu_+ - \frac{k}{2}.$$

(ii) *There exists $\nu_0 = \nu_0(\theta, \text{data}) \in (0, 1)$ such that, if*

$$(4.7) \quad |Q_\rho \cap \{n < \mu_- + k\}| \leq \nu_0 |Q_\rho|,$$

then it holds

$$(4.8) \quad \operatorname{ess\,inf}_{Q_{\rho/2}} n(x, t) \geq \mu_- + k/2.$$

Proof. First, we consider a level set $(n - \mu_+ + k)_+$ for a nonnegative bounded weak solution n of (1.2). We constructs sequences $\{\rho_i\}$, $\{k_i\}$, $\{K_i\}$, and $\{Q_i\}$ such that

$$(4.9) \quad \rho_i = \frac{\rho}{2} + \frac{\rho}{2^{i+1}}, \quad \rho_0 = \rho, \quad \rho_\infty = \frac{\rho}{2},$$

$$(4.10) \quad k_i = \frac{k}{2} + \frac{k}{2^{i+1}}, \quad k_0 = k, \quad k_\infty = \frac{k}{2},$$

$$(4.11) \quad K_i = K_{\rho_i}, \quad Q_i = K_i \times [-\theta k^{-\alpha} \rho_i^2, 0].$$

Moreover, we take a sequence of piecewise linear cutoff functions $\{\zeta_i\}_{i=0}^\infty$ such that

$$\zeta_i = \begin{cases} 1 & \text{in } Q_{i+1} \\ 0 & \text{on the parabolic boundary of } Q_i, \end{cases}$$

satisfying

$$|\nabla \zeta_i| \leq \frac{2^{i+2}}{\rho}, \quad \partial_t \zeta_i \leq \frac{2^{i+2} k^\alpha}{\theta \rho^2}.$$

It is easy to observe $(n - \mu_+ + k_i)_+ \leq k_i \leq k$ and $n > \mu_+ - k_i > \gamma k$ by choosing $k > (\gamma + 1/2)^{-1} \mu_+$ for some constant $\gamma > 0$.

Then the energy estimate given in Proposition 4.1 provides

$$\begin{aligned}
(4.12) \quad & \sup_{-\theta k^{-\alpha} \rho_i^2 < t < 0} \int_{K_i \times \{t\}} (n - \mu_+ + k_i)_+^2 \zeta_i^2 dx \\
& + C(\alpha, \gamma) k^\alpha \iint_{Q_i} |\nabla(n - \mu_+ + k_i)_+|^2 \zeta_i^2 dx dt \\
& \leq \left[\frac{2^{i+3}}{\theta} + 16(1 + \alpha)\gamma^\alpha \right] \frac{k^{2+\alpha}}{\rho^2} |Q_i \cap \{(n - \mu_+ + k_i)_+ > 0\}| \\
& + C(k^{2-\alpha} + k^2) \left[\int_{-\theta k^{-\alpha} \rho_i^2}^0 \left[A_{k_i, \rho_i}^+(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}.
\end{aligned}$$

Now we take the change of variable that $\bar{t} = k^\alpha t \in [-\theta \rho^2, 0]$. Also denote $\bar{n} = n(\cdot, \bar{t})$, $\bar{\zeta} = \zeta(\cdot, \bar{t})$, and $\bar{Q}_i = K_i \times [-\theta \rho_i^2, 0]$. Then (4.12) gives

$$\begin{aligned}
(4.13) \quad & \sup_{-\theta \rho_i^2 < \bar{t} < 0} \int_{K_i \times \{\bar{t}\}} (\bar{n} - \mu_+ + k_i)_+^2 \bar{\zeta}_i^2 dx \\
& + C(\alpha, \gamma) \iint_{\bar{Q}_i} |\nabla(\bar{n} - \mu_+ + k_i)_+|^2 \bar{\zeta}_i^2 dx d\bar{t} \\
& \leq \left[\frac{2^{i+3}}{\theta} + 16(1 + \alpha)\gamma^\alpha \right] \frac{k^2}{\rho^2} |\bar{Q}_i \cap \{(\bar{n} - \mu_+ + k_i)_+ > 0\}| \\
& + C(k^{2-\alpha} + k^2) k^{-\frac{2\alpha(1+\kappa)}{q_2}} \left[\int_{-\theta \rho_i^2}^0 \left[A_{k_i, \rho_i}^+(\bar{t}) \right]^{\frac{q_2}{q_1}} d\bar{t} \right]^{\frac{2(1+\kappa)}{q_2}}.
\end{aligned}$$

For simplicity, denote two sets

$$\begin{aligned}
A_i &= Q_i \cap \{(n - \mu_+ + k_i)_+ > 0\}, \\
\bar{A}_i &= \bar{Q}_i \cap \{(\bar{n} - \mu_+ + k_i)_+ > 0\}.
\end{aligned}$$

To handle the left hand side of (4.13), we apply Sobolev embedding theorem, Theorem 2.2, from which we calculate

$$\begin{aligned}
(4.14) \quad & \iint_{\bar{Q}_{i+1}} (\bar{n} - \mu_+ + k_i)_+^2 dx d\bar{t} \leq \iint_{\bar{Q}_i} (\bar{n} - \mu_+ + k_i)_+^2 \bar{\zeta}_i^2 dx d\bar{t} \\
& \leq |\bar{A}_i|^{\frac{2}{d+2}} \left[\sup_{-\theta \rho_i^2 < \bar{t} < 0} \int_{K_i \times \{\bar{t}\}} (\bar{n} - \mu_+ + k_i)_+^2 \bar{\zeta}_i^2 dx \right. \\
& \quad \left. + C(\alpha, \gamma) \iint_{\bar{Q}_i} |\nabla[(\bar{n} - \mu_+ + k_i)_+ \bar{\zeta}_i]|^2 dx d\bar{t} \right].
\end{aligned}$$

In the set $\{(\bar{n} - \mu_+ + k_{i+1})_+ > 0\}$, we observe that

$$(\bar{n} - \mu_+ + k_i)_+ \geq k_i - k_{i+1} = \frac{k}{2^{i+2}}.$$

Then the combination of (4.13) and (4.14), carrying cancellation on k^2 , we are able to say that

$$(4.15) \quad \begin{aligned} |\bar{A}_{i+1}| &\leq C(\alpha, d, \gamma, \theta) \frac{2^{3i}}{\rho^2} |\bar{A}_i|^{1+\frac{2}{d+2}} \\ &+ C(k^{-\alpha} + 1) k^{-\frac{2\alpha(1+\kappa)}{q_2}} \left[\int_{-\theta\rho_i^2}^0 \left[A_{k_i, \rho_i}^+(\bar{t}) \right]^{\frac{q_2}{q_1}} d\bar{t} \right]^{\frac{2(1+\kappa)}{q_2}} |\bar{A}_i|^{\frac{2}{d+2}}. \end{aligned}$$

We note that

$$\rho^2 \sim |\bar{Q}_i|^{\frac{2}{d+2}}, \quad |K_i| \sim |\bar{Q}_i|^{\frac{d}{d+2}}.$$

Let

$$\bar{Z}_i = \frac{1}{|K_i|} \left[\int_{-\theta\rho_i^2}^0 \left[A_{k_i, \rho_i}^+(\bar{t}) \right]^{\frac{q_2}{q_1}} d\bar{t} \right]^{\frac{2}{q_2}}.$$

By dividing (4.15) by $|\bar{Q}_i| = c(d)|\bar{Q}_{i+1}|$, we obtain (by applying (3.2))

$$(4.16) \quad \frac{|\bar{A}_{i+1}|}{|\bar{Q}_{i+1}|} \leq C(\alpha, d, \gamma, \theta) 2^{3i} \left\{ \frac{|\bar{A}_i|}{|\bar{Q}_i|} \right\}^{1+\frac{2}{d+2}} + C \bar{Z}_i^{1+\kappa} \left\{ \frac{|\bar{A}_i|}{|\bar{Q}_i|} \right\}^{\frac{2}{d+2}}.$$

Therefore, we take change variable back to t from \bar{t} from the dimensionless inequality (4.16), by letting $Y_i = |A_i|/|Q_i|$, we obtain the following inequality

$$(4.17) \quad Y_{i+1} \leq C_0 2^{3i} Y_i^{1+\frac{2}{d+2}} + C_1 2^{2i} Z_i^{1+\kappa} Y_i^{\frac{2}{d+2}}.$$

Then we are able to apply Lemma 2.5 that there exists

$$\nu_0 \leq (2C)^{-\frac{1+\kappa}{\sigma}} 2^{-\frac{3(1+\kappa)}{\sigma^2}}$$

where $C = \max\{C_0, C_1\}$ and $\sigma = \min\{\kappa, \frac{2}{d+2}\}$ such that Y_i and Z_i converge to 0 as $i \rightarrow \infty$. Hence, n is greater than $\mu_+ - \frac{k}{2}$ in almost everywhere of the set $Q_{\rho/2}$.

Second, we now carry the DeGiorgi iteration with the level sets $(n - \mu_- - k)_-$. It is easy to see that $0 \leq (n - \mu_- - k_i)_- < k_i \leq k$. We wish to avoid when μ_- is near zero so it is hard to estimate the lower bound of the local energy estimate (4.2). Therefore, we introduce

$$(4.18) \quad m = \max \left\{ n, \mu_- + \frac{k}{4} \right\}.$$

which provides that

$$(4.19) \quad \iint_{Q_i} m^\alpha |\nabla(m - \mu_- - k_i)_-|^2 \zeta_i^2 dx dt \leq \iint_{Q_i} n^\alpha |\nabla(n - \mu_- - k_i)_-|^2 \zeta_i^2 dx dt$$

considering two sets where $\{Q_i : m = n\}$ and $\{Q_i : m = \mu_- + \frac{k}{4}\}$ where $|\nabla(m - \mu_- - k_i)_-| = 0$ in the latter set. Moreover, we compute that

$$(4.20) \quad \begin{aligned} &\sup_{-\theta k^{-\alpha} \rho_i^2 < t < 0} \int_{K_i \times \{t\}} (m - \mu_- - k_i)_-^2 \zeta_i^2 dx \\ &\leq \sup_{-\theta k^{-\alpha} \rho_i^2 < t < 0} \int_{K_i \times \{t\}} (n - \mu_- - k_i)_-^2 \zeta_i^2 dx + \frac{\theta}{4^2} k^{2+\alpha} \rho_i^{-2} |Q_i \cap \{(n - \mu_- - k_i)_- > 0\}|. \end{aligned}$$

Then the combination of two inequalities (4.19) and (4.20) provides the energy estimates (4.12) in terms of m on the left-hand-side. By taking DeGiorgi iteration, it provides the existence of ν_0 such that if $|Q_\rho \cap \{n < \mu_- + k\}| \leq \nu_0 |Q_\rho|$, then it holds

$$\operatorname{ess\,inf}_{Q_{\rho/2}} m \geq \mu_- + \frac{k}{2},$$

which leads our conclusion. \square

Next proposition is a variant of DeGiorgi iteration using the information at a certain fixed time level to obtain the same conclusion as in Proposition 4.3 (in this proposition ν_0 is depending on both data and θ) where ν^* depending only on data.

Proposition 4.4. *Let n be a bounded nonnegative weak solution of (1.2) under (1.3) with $\alpha \geq 0$. For given positive constants ρ , k , and θ satisfying (3.2), let Q_ρ and $Q_{\rho/2}$ be given in Proposition 4.3.*

(i) *There exists $\nu^* \in (0, 1)$ determined only by data, such that, if*

$$n(x, -\theta k^{-\alpha} \rho^2) \leq \mu_+ - k,$$

and if

$$(4.21) \quad |Q_\rho \cap \{n > \mu_+ - k\}| \leq \frac{\nu^*}{\theta} |Q_\rho|,$$

then it holds

$$(4.22) \quad \operatorname{ess\,sup}_{Q_{\rho/2}} n(x, t) \leq \mu_+ - \frac{k}{2}.$$

(ii) *There exists $\nu^* \in (0, 1)$ determined only by data, such that, if*

$$n(x, -\theta k^{-\alpha} \rho^2) \geq \mu_- + k,$$

and if

$$(4.23) \quad |Q_\rho \cap \{n < \mu_- + k\}| \leq \nu_0 |Q_\rho|,$$

then it holds

$$(4.24) \quad \operatorname{ess\,inf}_{Q_{\rho/2}} n(x, t) \geq \mu_- + k/2.$$

The proof is done by repeating the same proof for Proposition 4.3 with $\frac{\nu^*}{\theta}$ instead of ν_0 . We refer Proposition 4.5 from [32].

4.3. The expansion of positive data. This section is to understand the behavior of a nonnegative bounded weak solution of (1.2) explaining positive data's flows in time and spatial axis sperately in measure sense (so called the expansion of positivities). The following proposition shows that if a nonnegative function is large on part of a cylinder, then it keeps largeness on part of a suitable time slice same as Proposition 4.1 in [32].

Proposition 4.5. *Let k , ρ , and T be positive constants. If n is a measurable nonnegative function defined on $Q = K_\rho \times (-T, 0)$ and if there is a constant $\nu_1 \in (0, 1)$ such that*

$$|Q \cap \{n > \mu_+ - k\}| \leq (1 - \nu_1) |Q|$$

there there is a number $\tau_1 \in (-T, -\frac{\nu_1}{2-\nu_1}T)$ for which

$$|\{x \in K_\rho : n(x, \tau_1) > \mu_+ - k\}| \leq (1 - \frac{\nu_1}{2}) |K_\rho|.$$

With the aid of logarithmic energy estimate in Proposition 4.2, we are able to control the measure where a weak solution keeps its largeness in a certain later time when we have according measure information at a fixed time level.

Proposition 4.6. *Let n be a nonnegative weak solution of (1.2) with $\alpha \geq 0$ under (1.3). Suppose that positive constants ρ , k , and $\mu \in (0, 1)$ are given satisfying (3.2).*

(i) *Assume that*

$$(4.25) \quad |\{x \in K_\rho : n(x, t_0) > \mu_+ - k\}| \leq (1 - \nu)|K_\rho|.$$

Then for any $\epsilon \in (0, 1)$, if $t_1 - t_0 \leq \theta k^{-\alpha} \rho^2$, there exist δ depending on the data, ν , and ϵ such that

$$(4.26) \quad |\{x \in K_\rho : n(x, t) > \mu_+ - \delta k\}| < (1 - (1 - \epsilon)\nu)|K_\rho|$$

for any $t_0 \leq t \leq t_1$.

(ii) *Assume that*

$$(4.27) \quad |\{x \in K_\rho : n(x, t_0) < \mu_- + k\}| \leq (1 - \nu)|K_\rho|.$$

Then for any $\epsilon \in (0, 1)$, if $t_1 - t_0 \leq \theta k^{-\alpha} \rho^2$, there exist δ depending on the data, ν , and ϵ such that

$$(4.28) \quad |\{x \in K_\rho : n(x, t) < \mu_- + \delta k\}| < (1 - (1 - \epsilon)\nu)|K_\rho|$$

for any $t_0 \leq t \leq t_1$.

Proof. We apply the logarithmic energy estimates from Proposition 4.2. Let ζ be a linear cutoff function independent of the time variable such that

$$\zeta = \begin{cases} 1 & \text{in } K_{(1-\sigma)\rho} \times [t_0, t], \\ 0 & \text{on the lateral boundary of } K_\rho \times [t_0, t] \end{cases}$$

for any $t \in (t_0, t_1)$ satisfying $|\nabla \zeta| \leq \frac{1}{\sigma \rho}$ and $\zeta_t = 0$ where $\sigma \in (0, 1)$ is to be determined later. Let $\delta = 2^{-j}$ for j is a positive integer which will be chosen later large enough. For $\Psi_\pm(n)$ in (4.3), we observe first, $(n - \mu_\pm \pm k)_\pm \leq k$ which provides

$$\Psi_\pm(n) \leq \ln \frac{1}{\delta} = j \ln 2 \quad \text{and} \quad |\Psi'_\pm(n)| \leq \frac{1}{\delta k}.$$

Moreover, in the set $\{n > \mu_+ - \delta k\}$ and $\{n < \mu_- + \delta k\}$ respectively, we have

$$(n - \mu_\pm \pm k)_\pm > (1 - \delta)k.$$

In both cases, it gives that

$$\Psi_\pm(n) \geq \ln \frac{1}{2\delta} = (j - 1) \ln 2.$$

From (4.4) and (3.2), it follows

$$(4.29) \quad \begin{aligned} & (j - 1)^2 (\ln 2)^2 |\{K_{(1-\sigma)\rho} : (n(\cdot, t) - \mu_\pm \pm \delta k)_\pm > 0\}| \\ & \leq j^2 (\ln 2)^2 |\{K_\rho : (n(\cdot, t_0) - \mu_\pm \pm k)_\pm > 0\}| \\ & \quad + C(\alpha, \gamma) \frac{k^\alpha j \ln 2}{\sigma^2 \rho^2} |t - t_0| |K_\rho| + C(\theta, \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2, \|\nabla B\|_{\hat{q}_1, \hat{q}_2}) (j \ln 2)^2 |K_\rho|. \end{aligned}$$

Hence all estimates above yields the following inequality:

$$(4.30) \quad \begin{aligned} & |\{K_{(1-\sigma)\rho} : (n(\cdot, t) - \mu_{\pm} \pm \delta k)_{\pm} > 0\}| \\ & \leq \left[\left(\frac{j}{j-1} \right)^2 (1 - \nu) + \frac{C(c_0, \alpha)j}{\sigma^2(j-1)^2} + C(\delta) \right] |K_{\rho}| \end{aligned}$$

by assumptions. Therefore, it leads that

$$(4.31) \quad \begin{aligned} & |\{K_{\rho} : (n(\cdot, t) - \mu_{\pm} \pm \delta k)_{\pm} > 0\}| \\ & \leq \left[\left(\frac{j}{j-1} \right)^2 (1 - \nu) + \frac{C(c_0, \alpha)j}{\sigma^2(j-1)^2} + C(\delta) + d\sigma \right] |K_{\rho}|. \end{aligned}$$

Then next we make a choice of j (so $\delta = 2^{-j}$) satisfying the following inequalities:

$$\left(\frac{j}{j-1} \right)^2 \leq 1 + \epsilon\nu, \quad \frac{C(c_0, \alpha)j}{\sigma^2(j-1)^2} \leq \frac{\epsilon\nu^2}{4}, \quad C(\delta) \leq \frac{\epsilon\nu^2}{4}, \quad d\sigma \leq \frac{\epsilon\nu^2}{4},$$

which provides (4.26) and (4.28). Let us choose

$$\sigma = \frac{\epsilon\nu^2}{4d}, \quad j = \max \left\{ 1 + \frac{1}{\sqrt{1 + \epsilon\nu} - 1}, \frac{C4^3 d^2 (1 + \epsilon\nu)}{\epsilon^3 \nu^6} \right\}.$$

The inequality for $C(\delta)$ is trivially following. \square

The following proposition is obtaining an arbitrary control over the measure of a cylinder where a weak solution is larger than some constant, if at each time level we know the measure of the set where a weak solution is somewhat large.

Proposition 4.7. *Let k , ρ , and θ be positive constants satisfying (3.2). Suppose that n is a nonnegative bounded weak solution of (1.2) under (1.3) with $\alpha \geq 0$ in $K_{2\rho} \times [-2\tau, 0]$. Then for any β and ν in $(0, 1)$, there exists $\delta^* = \delta^*(\beta, \nu, \theta, \text{data}) \in (0, 1)$ depending on data such that, if*

$$(4.32) \quad \tau \geq \theta(\delta^* k)^{-\alpha} \rho^2.$$

and if

$$(4.33) \quad |\{x \in K_{2\rho} : n(x, t) > \mu_+ - k\}| \leq (1 - \beta)|K_{2\rho}|$$

for all $t \in (-2\tau, 0]$, then we have

$$(4.34) \quad |\{(x, t) \in K_{\rho} \times [-\tau, 0] : n > \mu_+ - \delta^* k\}| \leq \nu |K_{\rho} \times [-\tau, 0]|.$$

Proof. Let $k_j = 2^{-j}k$ for $j = 0, 1, 2, \dots, j^*$ with j^* to be determined later. Denote that $\delta^* = 2^{-j^*}$. For simplicity, denote that

$$A_j = \{(x, t) \in K_{\rho} \times [-\tau, 0] : n(x, t) < k_j\}.$$

Let ζ be a poeewise linear cutoff function

$$\zeta = \begin{cases} 1, & \text{in } K_{\rho} \times [-\tau, 0] \\ 0, & \text{on the parabolic boundary of } K_{2\rho} \times [-2\tau, 0]. \end{cases}$$

satisfying, for all $j = 1, \dots, j^*$,

$$|\nabla \zeta| \leq \frac{1}{\rho}, \quad \text{and } \zeta_t \leq \frac{1}{\tau} = \frac{1}{\theta} (\delta^* k)^{\alpha} \rho^{-2} \leq \frac{1}{\theta} k_j^{\alpha} \rho^{-2}.$$

Then the local energy estimate Proposition 4.1 provides the following:

$$\begin{aligned}
(4.35) \quad & (1 + \alpha) \int_{-2\tau}^0 \int_{K_{2\rho}} n^\alpha |\nabla(n - \mu_+ + k_j)_+ \zeta|^2 dx dt \\
& \leq 2(1 + \alpha) \int_{-2\tau}^0 \int_{K_{2\rho}} k_j^\alpha (n - \mu_+ + k_j)_+^2 \zeta \partial_t \zeta dx dt \\
& \quad + 16(1 + \alpha) \int_{-2\tau}^0 \int_{K_{2\rho}} (n^\alpha + k_j^\alpha) (n - \mu_+ + k_j)_+^2 |\nabla \zeta|^2 dx dt \\
& \quad + 16k^2 (k^{-\alpha} \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 + \|\nabla B\|_{\hat{q}_1, \hat{q}_2}) |2\tau|^{\frac{2(1+\kappa)}{q_2}} |K_{2\rho}|^{\frac{2(1+\kappa)}{q_1}}
\end{aligned}$$

In the set $\{(n - \mu_+ + k_j)_+ > 0\}$, we observe that $n > \mu_+ - k_j$. Because of (3.2), the last term is simplifies as

$$16 \frac{k^{2-\alpha}}{\rho^2} (k^{-\alpha} \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 + \|\nabla B\|_{\hat{q}_1, \hat{q}_2}) |2\tau|^{\frac{2(1+\kappa)}{q_2}} |K_{2\rho}|^{\frac{2(1+\kappa)}{q_1}} \leq C \frac{k^2}{\rho^2} |K_{2\rho} \times [-2\tau, 0]|.$$

Then the inequality (4.35) yields

$$(4.36) \quad \int_{-2\tau}^0 \int_{K_{2\rho}} |\nabla(n - \mu_+ + k_j)_+ \zeta|^2 dx dt \leq C_0 \frac{k_j^2}{\rho^2} |K_{2\rho} \times [-2\tau, 0]|$$

where $C_0 = C_0(\alpha, \theta, \gamma, \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2, \|\nabla B\|_{\hat{q}_1, \hat{q}_2})$.

By taking integration of the obtained inequality obtained from Lemma 2.6 for all $t \in [-\tau, 0]$, we have

$$(4.37) \quad 2^{-2} k_j^2 |K_\rho \times [-\tau, 0] \cap \{n > \mu_- - k_j\}| \leq \frac{\gamma\rho}{\beta} \iint_{A_j \setminus A_{j+1}} |\nabla(n - \mu_+ + k_j)_+| dx dt.$$

The inequality (4.33) yields the following inequality for all $t \in (-2\tau, 0)$

$$|K_\rho \cap \{\mu_+ - k_j < n < \mu_+ - k_{j+1}\}| \geq \beta |K_{2\rho}|.$$

For the simplicity, let $\Omega_\tau = K_\rho \times [-\tau, 0]$. Let us divide (4.37) by $|A_j \setminus A_{j+1}|$ and apply Jensen's inequality to obtain the following inequality:

$$\begin{aligned}
(4.38) \quad & k_j^{2+2\alpha} \left(\frac{|A_j|}{|A_j \setminus A_{j+1}|} \right)^2 \leq \left(\frac{\gamma\rho}{\beta} \right)^2 \frac{1}{|A_j \setminus A_{j+1}|} \iint_{A_j \setminus A_{j+1}} |\nabla(n - \mu_+ + k_j)_+|^2 dx dt \\
& \leq C_1(\alpha, \theta, \gamma, \beta) k_j^{2+2\alpha} \frac{|\Omega_\tau|}{|A_j \setminus A_{j+1}|}.
\end{aligned}$$

Hence, it yields constant C_2 independent of j satisfying

$$(4.39) \quad \left(\frac{|A_j|}{|A_j \setminus A_{j+1}|} \right)^2 \leq C_2 \frac{|\Omega_\tau|}{|A_j \setminus A_{j+1}|}.$$

Therefore, (4.39) provides that

$$(4.40) \quad \left(\frac{|A_{j+1}|}{|\Omega_\tau|} \right)^2 = \left(\frac{|A_{j+1}|}{|A_j \setminus A_{j+1}|} \right)^2 \left(\frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|} \right)^2 \leq C_2 \frac{|A_j \setminus A_{j+1}|}{|\Omega_\tau|}.$$

Then by taking the sum over $j = 0, 1, \dots, j^* - 1$, we obtain

$$j^* \left(\frac{|A_{j^*}|}{|\Omega_\tau|} \right)^2 \leq C_2 \frac{|A_0 \setminus A_{j^*}|}{|\Omega_\tau|} \leq C_2.$$

By choosing j^* large enough such that $j^* \geq C_2 \nu^{-2}$, it leads that $|A_{j^*}| \leq \nu |\Omega_\tau|$. \square

4.4. Proof of two alternatives. First, we provide the proof of Lemma 3.1.

Proof of Lemma 3.1: Because of the assumption (3.4), we apply Proposition 4.3 with $\theta = 1$, $k = \omega/2$, and $\rho = R$ to infer that

$$n \geq \mu_- + \frac{\omega}{4} \quad \text{on } K_R \times \{T_1\}$$

where

$$T_1 = T_0 + \Delta - \left(\frac{\omega}{2}\right)^{-\alpha} R^2.$$

Therefore, it holds, for $\nu = 1$,

$$\left| \{K_R : n(x, T_1) < \mu_- + \frac{\omega}{4}\} \right| \leq (1 - \nu) |K_R|.$$

By applying Proposition 4.6 with $\rho = R$, $\nu = 1$, $t_0 = T_1$, $\theta = 1$ and $k = \omega/4$, for any $\epsilon \in (0, 1)$, there exists $\delta = \delta(\text{data}, \epsilon, \theta_1)$ satisfying

$$(4.41) \quad \left| \{K_R : n(x, t) < \mu_- + \frac{\delta\omega}{4}\} \right| \leq (1 - (1 - \epsilon)\nu) |K_R|$$

for all

$$t \geq T_1 + \left(\frac{\omega}{4}\right)^{-\alpha} R^2.$$

Let us choose

$$\theta_1 = \theta_0 2^{2-\alpha} - 3 \cdot 2^{-\alpha}$$

which provides that (4.41) holds for all time $[-T_1, 0]$.

Let us choose $\epsilon = \nu^*/(2^{2-\alpha}\theta_0)$ from Proposition 4.4 and set

$$\theta = \frac{-T_1}{(\delta\omega/4)^{-\alpha} R^2}.$$

We observe that $\theta \leq 2^{2-\alpha}\theta_0$. Hence, Proposition 4.4 provides that

$$\text{ess inf}_Q n(x, t) \geq \mu_- + \frac{\delta\omega}{8}$$

where

$$Q = K_{R/2} \times \left[-\theta \left(\frac{\delta\omega}{4}\right)^{-\alpha} \left(\frac{R}{2}\right)^2, 0 \right].$$

We make conclusion by choosing $\theta = 4^{-\alpha}\omega^\alpha$ and $\theta_0 = 2^{-2-\alpha}\delta^\alpha + 1$.

Now, we provide the proof of Lemma 3.2.

Proof of Lemma 3.2: From the assumption (3.6), we apply Proposition 4.5 with $k = \omega/2$, $\nu_1 = \nu_0$, and $\rho = 2R$, that there exists

$$\tau_1 \in \left(T_0, T_0 + \frac{\nu_0}{2 - \nu_0} \Delta \right)$$

such that

$$\left| K_{2R} \cap \{n(x, \tau_1) > \mu_+ - \frac{\omega}{2}\} \right| \leq \left(1 - \frac{\nu_0}{2}\right) |K_{2R}|.$$

We apply Proposition 4.6 with $\rho = 2R$, $k = \omega/2$, $\epsilon = 1/2$ and $\theta = \theta_0$. Then there exists δ depending on data such that

$$(4.42) \quad \left| K_{2R} \cap \{n(x, t) > \mu_+ - \frac{\delta\omega}{2}\} \right| \leq \left(1 - \frac{\nu_0}{4}\right) |K_{2R}| \quad \text{for all } t \in [\tau_1, 0]$$

because $\tau_1 + \theta_0 \Delta \geq 0$.

We are ready to apply Proposition 4.7 with $k = \delta\omega/2$, $\theta = \theta_0 - 1$, $\beta = \nu_0/4$ and $\rho = R$. Then for any $\nu \in (0, 1)$, there exists $\delta^* = \delta^*(\text{data}, \nu)$ such that

$$\left| K_R \times [T, 0] \cap \left\{ n > \mu_+ - \frac{\delta^* \delta \omega}{2} \right\} \right| \leq \nu^* |K_R \times [T, 0]|$$

where

$$T = -(\theta_0 - 1) \left(\frac{\delta^* \delta \omega}{2} \right)^{-\alpha} R^2,$$

by determining $\theta_0 = 1 + (\delta^* \delta)^{-\alpha}$. Let us choose $\nu = \nu_0$, the constant from Proposition 4.4 which yields

$$\text{ess sup}_Q n(x, t) \leq \mu_+ - \frac{\delta^* \delta \omega}{4}$$

where

$$Q = K_{R/2} \times \left[-\left(\frac{\omega}{2} \right)^{-\alpha} \left(\frac{R}{2} \right)^2, 0 \right].$$

5. PROOFS OF THEOREM 1.7 AND THEOREM 1.8

We introduce the approximate system of (1.1), which is given by

$$(5.1) \quad \begin{cases} \partial_t n_\epsilon - \Delta(n_\epsilon + \epsilon)^{1+\alpha} + u_\epsilon \cdot \nabla n_\epsilon = -\nabla \cdot (\chi(c_\epsilon)(n_\epsilon + \epsilon)^q \nabla c_\epsilon), \\ \partial_t c_\epsilon - \Delta c_\epsilon + u_\epsilon \cdot \nabla c_\epsilon = -\kappa(c_\epsilon) n_\epsilon, \\ \partial_t u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = -n_\epsilon \nabla \phi, \\ \nabla \cdot u_\epsilon = 0, \end{cases}$$

in $\mathbb{R}^3 \times (0, T)$ with smooth initial data $(n_{0_\epsilon}, c_{0_\epsilon}, u_{0_\epsilon})$ defined by

$$n_{0_\epsilon} = \psi_\epsilon * n_0, \quad c_{0_\epsilon} = \psi_\epsilon * c_0 \quad \text{and} \quad u_{0_\epsilon} = \psi_\epsilon * u_0,$$

where ψ_ϵ denotes the usual mollifier with $\epsilon \in (0, 1)$ and $*$ denotes the space convolution.

It is known that, due to the standard theory of existence and regularity as done in [22] and [54], there exists a classical solution of the equation (5.1) locally in time for each $\epsilon \in (0, 1)$. In the sequel, although we have to obtain estimates from the approximate system (5.1), we compute a priori estimates only for simplicity, since it turns out that all computations are independent of ϵ .

Now we present the proof of Theorem 1.7.

Proof of Theorem 1.7.

We consider first the case that as $\frac{9q-8}{6} < \alpha \leq \frac{3q-2}{2}$ and the other case $\alpha \geq \frac{3q-2}{2}$ will be treated later.

• (Case; $\frac{9q-8}{6} < \alpha \leq \frac{3q-2}{2}$) Multiplying equation (1.1)₁ with $\log n$ and using the integration by parts, we have

$$(5.2) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} n \log n + \frac{4}{1+\alpha} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 = \frac{1}{q} \int_{\mathbb{R}^3} \nabla n^q \chi(c) \nabla c \\ & \leq \frac{2\bar{\chi}}{2\alpha - q + 2} \left(\int_{\mathbb{R}^3} |\nabla n^{\frac{2\alpha-q+2}{2}}| n^{\frac{3q-2\alpha-2}{2}} |\nabla c| \right) := J_1, \end{aligned}$$

where $\overline{\chi} := \max_{\mathbb{R}_T^3} |\chi(c(\cdot))|$. Here we remark that the restriction that $\alpha \leq \frac{3q-2}{2}$ is due to the requirement that $3q - 2\alpha - 2 \geq 0$ in (5.2). Applying Young's inequality,

$$(5.3) \quad J_1 \leq \varepsilon_1 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + C_{\varepsilon_1} \int_{\mathbb{R}^3} n^{3q-2\alpha-2} |\nabla c|^2,$$

where ε_1 is a sufficiently small constant, which will be chosen later. In the sequel, we indicate ε_i , $i = 1, 2, \dots$ as a small constant, which will be decided later. Combining the above estimates, we obtain

$$(5.4) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} n \log n + \frac{4}{1+\alpha} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 \\ & \leq \varepsilon_1 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + C_{\varepsilon_1} \int_{\mathbb{R}^3} n^{3q-2\alpha-2} |\nabla c|^2. \end{aligned}$$

Next, testing $n^{\alpha-q+1}$ to (1.1)₁ and using Hölder and Young's inequalities,

$$(5.5) \quad \begin{aligned} & \frac{1}{\alpha-q+2} \frac{d}{dt} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \frac{4(1+\alpha)(\alpha-q+1)}{(2\alpha-q+2)^2} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \\ & \leq \frac{\alpha-q+1}{1+\alpha} \overline{\chi} \int_{\mathbb{R}^3} |\nabla n^{\frac{2\alpha-q+2}{2}}| n^{\frac{q}{2}} |\nabla c| \\ & \leq \varepsilon_2 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + C_{\varepsilon_2} \int_{\mathbb{R}^3} n^q |\nabla c|^2. \end{aligned}$$

We estimate second term in righthand side of (5.5) via integration by parts.

$$(5.6) \quad \begin{aligned} & \int_{\mathbb{R}^3} n^q |\nabla c|^2 = \int_{\mathbb{R}^3} n^q \nabla c \cdot \nabla c \leq \int_{\mathbb{R}^3} |\nabla n^q| c |\nabla c| + \int_{\mathbb{R}^3} n^q c |\nabla^2 c| \\ & \leq C \int_{\mathbb{R}^3} |\nabla n^{\frac{2\alpha-q+2}{2}}| n^{\frac{3q-2\alpha-2}{2}} |\nabla c| + C \int_{\mathbb{R}^3} n^q |\nabla^2 c| \\ & \leq \varepsilon_3 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + C_{\varepsilon_3} \int_{\mathbb{R}^3} n^{3q-2\alpha-2} |\nabla c|^2 + C \int_{\mathbb{R}^3} n^q |\nabla^2 c|, \end{aligned}$$

where we used that $\|c(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|c_0\|_{L^\infty(\mathbb{R}^3)}$. Combining the estimates above and taking ε_i , $i = 1, 2, 3$ sufficiently small, we conclude that

$$(5.7) \quad \frac{d}{dt} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \leq C \left(\int_{\mathbb{R}^3} n^{3q-2\alpha-2} |\nabla c|^2 + \int_{\mathbb{R}^3} n^q |\nabla^2 c| \right).$$

Multiplying equation (1.1)₂ with $-\Delta c$ and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla c|^2 + \|\nabla^2 c\|_2^2 \leq \int_{\mathbb{R}^3} (u \cdot \nabla c) \nabla^2 c + \overline{\kappa} \int_{\mathbb{R}^3} n |\nabla^2 c|,$$

where $\overline{\kappa} := \max_{\mathbb{R}_T^3} |\kappa(c(\cdot))|$. Due to $\nabla \cdot u = 0$ and uniform bound of c , we observe that

$$\begin{aligned} \int_{\mathbb{R}^3} (u \cdot \nabla c) \Delta c &= \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_i c_{x_i} c_{x_j x_j} = - \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_{i, x_j} c_{x_i} c_{x_j} \\ &= \sum_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} u_{i, x_j} c c_{x_j x_i} \leq C \|\nabla u\|_2 \|\nabla^2 c\|_2. \end{aligned}$$

Summing up estimates, we obtain

$$(5.8) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla c|^2 + \|\nabla^2 c\|_2^2 \leq C \left(\|\nabla u\|_2 \|\nabla^2 c\|_2 + \int_{\mathbb{R}^3} n |\nabla^2 c| \right).$$

Let M be a sufficiently large positive constant, which will be specified later. Multiplying equation (1.1)₃ with Mu and using the integration by parts, we note that

$$(5.9) \quad \frac{M}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 + 2M \|\nabla u\|_2^2 \leq M \|\nabla \phi\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} n |u|.$$

We recall that $\int n |\log n|$, we recall that (see e.g., (18) of [12])

$$(5.10) \quad \int_{\mathbb{R}^3} n |\log n| \leq \int_{\mathbb{R}^3} n \log n + 2 \int_{\mathbb{R}^3} \langle x \rangle n + C$$

and we note that (see e.g., (19) of [12])

$$(5.11) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n \leq C \left(1 + \|\nabla c\|_2^2 + \|\nabla u\|_2^2 \right) + \varepsilon \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \varepsilon \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2.$$

Summing up (5.4)–(5.9) and (5.11), and taking sufficiently small ε_i , $i = 1, 2, 3$, we have

$$(5.12) \quad \begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}^3} n \log n + \int_{\mathbb{R}^3} \langle x \rangle n + \frac{1}{\alpha - q + 2} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla c|^2 + \frac{M}{2} \int_{\mathbb{R}^3} |u|^2 \right) \\ & + C \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\ & \leq C \left(\int_{\mathbb{R}^3} n^{3q-2\alpha-2} |\nabla c|^2 + \int_{\mathbb{R}^3} n^q |\nabla^2 c| \right. \\ & \quad \left. + \|\nabla u\|_2 \|\nabla^2 c\|_2 + \int_{\mathbb{R}^3} n |\Delta c| + M \int_{\mathbb{R}^3} n |u| \right) + C \left(1 + \|\nabla c\|_2^2 \right) \\ & = C (\text{I} + \text{II} + \text{III} + \text{IV} + \text{V}) + C \left(1 + \|\nabla c\|_2^2 \right). \end{aligned}$$

Due to $0 \leq 3q - 2\alpha - 2 < \frac{2}{3}$ via $\frac{9q-8}{6} < \alpha \leq \frac{3q-2}{2}$, we estimate I as follows:

$$(5.13) \quad \text{I} \leq \begin{cases} \|\nabla c\|_2^2, & \text{if } \alpha = \frac{3q-2}{2}, \\ C_{\varepsilon_4} \|\nabla c\|_2^2 + \varepsilon_4 \|n_0\|_1^{\frac{2}{3}} \|\nabla^2 c\|_2^2, & \text{if } \frac{9q-8}{6} < \alpha < \frac{3q-2}{2}, \end{cases}$$

where Hölder inequality and Sobolev embedding are used.

To estimate the term II, applying Hölder, Young's and interpolation inequalities, we have

$$\begin{aligned} \text{II} &= \int_{\mathbb{R}^3} n^q |\nabla^2 c| \leq C_{\varepsilon_5} \|n\|_{2q}^{2q} + \varepsilon_5 \|\nabla^2 c\|_2^2 \leq C_{\varepsilon_5} \|n\|_1^{2q\theta} \|n\|_{3(2\alpha-q+2)}^{2q(1-\theta)} + \varepsilon_5 \|\nabla^2 c\|_2^2 \\ &\leq C_{\varepsilon_5} \|n\|_1^{2q\theta} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2q(1-\theta) \frac{2}{2\alpha-q+2}} + \varepsilon_5 \|\nabla^2 c\|_2^2, \end{aligned}$$

where $\theta = \frac{6\alpha-5q+6}{2q(6\alpha-3q+5)}$. Since $\alpha > \frac{9q-8}{6}$, we observe that

$$2q(1-\theta) \frac{2}{2\alpha-q+2} = \frac{6(2q-1)}{6\alpha-3q+5} < 2.$$

Therefore, we have

$$(5.14) \quad \text{II} \leq C_{\varepsilon_5} C_{\varepsilon_6} + \varepsilon_6 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + \varepsilon_5 \|\nabla^2 c\|_2^2.$$

The term III is easily estimated as follows:

$$(5.15) \quad \text{III} = \|\nabla u\|_2 \|\nabla^2 c\|_2 \leq C_{\varepsilon_7} \|\nabla u\|_2^2 + \varepsilon_7 \|\nabla^2 c\|_2^2.$$

Next, we estimate the term IV. Hölder and Young's and interpolation inequalities yield

$$\begin{aligned} \text{IV} &= \int_{\mathbb{R}^3} n |\nabla^2 c| \leq C_{\varepsilon_8} \|n\|_1^{2\theta} \|n\|_{3(2\alpha-q+2)}^{2(1-\theta)} + \varepsilon_8 \|\nabla^2 c\|_2^2 \\ &\leq C_{\varepsilon_8} \|n\|_1^{2\theta} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-q+2}} + \varepsilon_8 \|\nabla^2 c\|_2^2, \end{aligned}$$

where $\theta = \frac{6\alpha-3q+4}{2(6\alpha-3q+5)}$. Similarly as above, we note, due to $\alpha > \frac{9q-8}{6}$, that

$$2(1-\theta) \frac{2}{2\alpha-q+2} = \frac{6}{6\alpha-3q+5} < 2.$$

Therefore, we have

$$(5.16) \quad \text{IV} \leq C_{\varepsilon_8} C_{\varepsilon_9} + \varepsilon_9 \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + \varepsilon_8 \|\nabla^2 c\|_2^2.$$

Finally, the term V is estimated via Hölder, Young's inequalities and Sobolev embedding. Indeed,

$$\begin{aligned} \text{V} &= M \int_{\mathbb{R}^3} n |u| \, dx \leq M \left(\|n\|_{\frac{6}{5}} \|u\|_6 \right) \leq M \left(C_{\varepsilon_{10}} \|n\|_{\frac{6}{5}}^2 + \varepsilon_{10} \|\nabla u\|_2^2 \right) \\ &\leq M \left(C_{\varepsilon_{10}} \|n\|_1^{2\theta} \|n\|_{3(2\alpha-q+2)}^{2(1-\theta)} + \varepsilon_{10} \|\nabla u\|_2^2 \right) \\ &\leq M \left(C_{\varepsilon_{10}} \|n\|_1^{2\theta} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-q+2}} + \varepsilon_{10} \|\nabla u\|_2^2 \right), \end{aligned}$$

where $\theta = \frac{10\alpha-5q+8}{2(6\alpha-3q+1)}$. As before, via $\alpha > \frac{9q-8}{6}$, we can see that

$$2(1-\theta) \frac{2}{2\alpha-q+2} = \frac{2}{6\alpha-3q+5} < 2.$$

Therefore, we obtain

$$(5.17) \quad \text{V} \leq M \left(C_{\varepsilon_{10}} C_{\varepsilon_{11}} + \varepsilon_{11} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + \varepsilon_{10} \|\nabla u\|_2^2 \right).$$

Summing up (5.13)–(5.17), we have for sufficiently small ε_i , $i = 4, \dots, 11$

$$\begin{aligned} (5.18) \quad &\sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n \log n + \int_{\mathbb{R}^3} \langle x \rangle n + \frac{1}{\alpha-q+2} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \right) \\ &+ C \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\ &\leq C \left(1 + \int_0^T \left(\|\nabla c\|_2^2 + 1 \right) \right) \leq C, \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \int_{\mathbb{R}^3} n_0 \log n_0, \|n_0\|_{\alpha-q+2}, \|n_0(1 + \langle x \rangle)\|_1, \|u_0\|_2 \right)$. Combining estimates (5.10) and (5.18)

$$\begin{aligned}
 (5.19) \quad & \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n |\log n| + \langle x \rangle n + \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\
 & + \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\
 & \leq C \left(1 + \int_0^T \left(\|\nabla c\|_2^2 + 1 \right) \right) \leq C,
 \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \|n_0(1 + |x| + |\log n_0|)\|_1, \|n_0\|_{\alpha-q+2}, \|u_0\|_2 \right)$.

• (Case $\alpha > \frac{3q-2}{2}$) Multiplying equation (1.1)₁ with $\log n$ and integrating it by parts, we have

$$\begin{aligned}
 (5.20) \quad & \frac{d}{dt} \int_{\mathbb{R}^3} n \log n + \frac{4}{1+\alpha} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 = \frac{1}{q} \int_{\mathbb{R}^3} \chi(c) \nabla n^q \cdot \nabla c, \\
 & = -\frac{1}{q} \int_{\mathbb{R}^3} (\chi'(c) n^q |\nabla c|^2 + \chi(c) n^q \nabla^2 c) \\
 & \leq \frac{1}{q} \int_{\mathbb{R}^3} (\chi'(c) n^q |\nabla c|^2 + \chi(c) n^q |\nabla^2 c|).
 \end{aligned}$$

Using estimates (5.5), (5.8), (5.9), (5.11) together with (5)

$$\begin{aligned}
 (5.21) \quad & \frac{d}{dt} \left(\int_{\mathbb{R}^3} n (\log n + 2\langle x \rangle) + \frac{1}{\alpha-q+2} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla c|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \right) \\
 & + C \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\
 & \leq C \left(\int_{\mathbb{R}^3} n^q |\nabla c|^2 + n^q |\nabla^2 c| + \|\nabla u\|_2 \|\nabla^2 c\|_2 + \int_{\mathbb{R}^3} n |\nabla^2 c| + \int_{\mathbb{R}^3} n |u| \right) + C \left(1 + \|\nabla c\|_2^2 \right) \\
 & = C (\text{I} + \text{II} + \text{III} + \text{IV} + \text{V}) + C \left(1 + \|\nabla c\|_2^2 \right).
 \end{aligned}$$

We estimate II, III, IV and V exactly the same ways as (5.14), (5.15), (5.16) and (5.17), respectively. It remains to estimate I. Due to $0 < q < \frac{2\alpha-q+2}{2}$ via $\alpha > \frac{3q-2}{2}$, we have

$$\text{I} \leq \int_{\mathbb{R}^3} \left(C_{\varepsilon_{12}} + \varepsilon_{12} n^{\frac{2\alpha-q+2}{2}} \right) |\nabla c|^2 \leq C_{\varepsilon_{12}} \|\nabla c\|_2^2 + \varepsilon_{12} \int_{\mathbb{R}^3} n^{\frac{2\alpha-q+2}{2}} |\nabla c|^2.$$

We note that

$$\begin{aligned}
 (5.22) \quad & \int_{\mathbb{R}^3} n^{\frac{2\alpha-q+2}{2}} |\nabla c|^2 \leq \int_{\mathbb{R}^3} |\nabla n^{\frac{2\alpha-q+2}{2}}| |\nabla c| + \int_{\mathbb{R}^3} n^{\frac{2\alpha-q+2}{2}} |\nabla^2 c| \\
 & \leq \frac{1}{2} \left(\left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + \|\nabla c\|_2^2 + \left\| n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right).
 \end{aligned}$$

The last term of the right hand side in (5.22) is estimated as follows:

$$\begin{aligned}
 & \left\| n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 = \|n\|_{2\alpha-q+2}^{2\alpha-q+2} \leq \|n\|_1^{(2\alpha-q+2)\theta} \|n\|_{3(2\alpha-q+2)}^{(2\alpha-q+2)(1-\theta)} \\
 & \leq C \|n\|_1^{(2\alpha-q+2)\theta} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{(2\alpha-q+2)(1-\theta) \frac{2}{2\alpha-q+2}},
 \end{aligned}$$

where $\theta = \frac{2}{6\alpha-3q+5}$. We then note that

$$(2\alpha - q + 2)(1 - \theta) \frac{2}{2\alpha - q + 2} = \frac{6(2\alpha - q + 1)}{6\alpha - 3q + 5} < 2.$$

Hence we obtain

$$(5.23) \quad \mathbf{I} \leq C \|\nabla c\|_2^2 + C\varepsilon_{12} \left(\|\nabla^2 c\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right).$$

Adding up estimates, we conclude that

$$(5.24) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n \log n + \int_{\mathbb{R}^3} \langle x \rangle n + \frac{1}{\alpha - q + 2} \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \right) \\ & + C \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\ & \leq C \left(1 + \int_0^T (\|\nabla c\|_2^2 + 1) \right) \leq C, \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \int_{\mathbb{R}^3} n_0 \log n_0, \|n_0\|_{\alpha-q+2}, \|n_0(1 + \langle x \rangle)\|_1, \|u_0\|_2 \right)$. Combining estimates (5.10) and (5.24)

$$(5.25) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n |\log n| + \int_{\mathbb{R}^3} \langle x \rangle n + \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\ & + \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\ & \leq C \left(1 + \int_0^T (\|\nabla c\|_2^2 + 1) \right) \leq C, \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \|n_0(1 + |x| + |\log n_0|)\|_1, \|n_0\|_{\alpha-q+2}, \|u_0\|_2 \right)$. This completes the proof. \square

We present proof of Theorem 1.8.

Proof of Theorem 1.8. Since we showed already the case $\alpha > \frac{9q-8}{6}$ in Theorem 1.7, it suffices to prove the case $2q-2 < \alpha \leq \frac{9q-8}{6}$, $1 \leq q \leq \frac{4}{3}$.

Multiplying equation (1.1)₁ with $\log n$ and using the integration by parts, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} n \log n + \frac{4}{1+\alpha} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 &= \frac{1}{q} \int_{\mathbb{R}^3} \nabla n^q \chi(c) \nabla c \\ &\leq \frac{2\overline{\chi}}{1+\alpha} \int_{\mathbb{R}^3} |\nabla c| |\nabla n^{\frac{1+\alpha}{2}}| n^{\frac{2q-\alpha-1}{2}}. \end{aligned}$$

where $\overline{\chi}$ denote $\max_{\mathbb{R}_T^3} |\chi(c(\cdot))|$. Applying Hölder and Young's inequalities, we have

$$\begin{aligned} & \frac{2\overline{\chi}}{1+\alpha} \int_{\mathbb{R}^3} |\nabla c| |\nabla n^{\frac{1+\alpha}{2}}| n^{\frac{2q-\alpha-1}{2}} \leq \int_{\mathbb{R}^3} |\nabla c| |\nabla n^{\frac{1+\alpha}{2}}| \left(C_{\varepsilon_1} + \varepsilon_1 n^{\frac{1}{2}} \right) \\ & \leq C_{\varepsilon_1} \left(C_{\varepsilon_2} \|\nabla c\|_2^2 + \varepsilon_2 \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 \right) + \varepsilon_1 \left(C_{\varepsilon_3} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \varepsilon_3 \left\| n^{\frac{1}{2}} \nabla c \right\|_2^2 \right). \end{aligned}$$

Hence we have

$$(5.26) \quad \frac{d}{dt} \int_{\mathbb{R}^3} n \log n + \left(\frac{4}{1+\alpha} - C_{\varepsilon_1} \varepsilon_2 - C_{\varepsilon_3} \varepsilon_1 \right) \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 \leq C_{\varepsilon_1} C_{\varepsilon_2} \|\nabla c\|_2^2 + \varepsilon_1 \varepsilon_3 \left\| n^{\frac{1}{2}} \nabla c \right\|_2^2,$$

where $2q - \alpha - 1 > 0$ and $\frac{2q-\alpha-1}{2} < \frac{1}{2}$, which is equivalent to $2q - 2 < \alpha \leq \frac{9q-8}{6}$, $1 \leq q \leq \frac{4}{3}$.

Multiplying equation (1.1)₁ with $n^{\alpha-2q+2}$ and using the integration by parts, we have

$$\begin{aligned} & \frac{1}{\alpha - 2q + 3} \frac{d}{dt} \int_{\mathbb{R}^3} |n|^{\alpha-2q+3} + \frac{4(1+\alpha)(\alpha-2q+2)}{(2\alpha-2q+3)^2} \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^2 \\ & \leq \frac{\alpha-2q+2}{\alpha-q+2} \bar{\chi} \int_{\mathbb{R}^3} |\nabla n^{\alpha-q+2}| \cdot |\nabla c|. \end{aligned}$$

Applying Hölder and Young's inequalities, we have

$$(5.27) \quad \frac{1}{\alpha - 2q + 3} \frac{d}{dt} \int_{\mathbb{R}^3} |n|^{\alpha-2q+3} + \left(\frac{4(1+\alpha)(\alpha-2q+2)}{(2\alpha-2q+3)^2} - \varepsilon_7 \right) \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^2 \leq C_{\varepsilon_7} \int_{\mathbb{R}^3} n |\nabla c|^2,$$

where $\alpha - 2q + 3 > 1$, which is equivalent to $\alpha > 2q - 2$. Multiplying equation (1.1)₂ with $-\nabla^2 c$ and using the integration by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla c|^2 + \|\nabla^2 c\|_2^2 \leq \int_{\mathbb{R}^3} (u \cdot \nabla c) \nabla^2 c + \int_{\mathbb{R}^3} \kappa(c) n |\nabla^2 c| \\ & = \text{I} + \text{II}. \end{aligned}$$

The term I is estimated as follows. Applying Hölder and Young's inequalities, we have

$$\text{I} = C_{\varepsilon_8} \|\nabla u\|_2^2 + \varepsilon_8 \|\nabla^2 c\|_2^2.$$

Using the integration by parts, we have

$$\text{II} = \int_{\mathbb{R}^3} \kappa(c) n |\nabla^2 c| = - \int_{\mathbb{R}^3} \kappa'(c) n |\nabla c|^2 + \int_{\mathbb{R}^3} \kappa(c) |\nabla n| \cdot |\nabla c|.$$

And using Hölder and Young's inequalities, we have

$$\begin{aligned} \text{II} & \leq -\kappa_0 \int_{\mathbb{R}^3} n |\nabla c|^2 + \int_{\mathbb{R}^3} \kappa(c) |\nabla n^{\frac{1+\alpha}{2}}| (C_{\varepsilon_9} + \varepsilon_9 n^{\frac{1}{2}}) |\nabla c| \\ & \leq -\kappa_0 \int_{\mathbb{R}^3} n |\nabla c|^2 + (\varepsilon_{10} + \varepsilon_9 \varepsilon_{11}) \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + C_{\varepsilon_8} C_{\varepsilon_9} \|\nabla c\|_2^2 + \varepsilon_9 C_{\varepsilon_{11}} \left\| n^{\frac{1}{2}} \nabla c \right\|_2^2. \end{aligned}$$

Hence we have

$$(5.28) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla c|^2 + (1 - \varepsilon_8) \|\nabla^2 c\|_2^2 + (\kappa_0 - \varepsilon_9 C_{\varepsilon_{11}}) \int_{\mathbb{R}^3} n |\nabla c|^2 \\ & \leq (\varepsilon_{10} + \varepsilon_9 \varepsilon_{11}) \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + C_{\varepsilon_8} \|\nabla c\|_2^2 + C_{\varepsilon_8} \|\nabla u\|_2^2, \end{aligned}$$

where $\frac{1}{1-\alpha} > 1$, which is equivalent to $0 < \alpha < 1$. Let M be a sufficiently large positive constant, which will be decided later. Multiplying equation (1.1)₃ with Mu and using the integration by parts, we have

$$\frac{M}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 + 2M \|\nabla u\|_2^2 \leq M \|\nabla \phi\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} n |u|.$$

Using interpolation and Young's inequalities, we have

$$\begin{aligned} \frac{M}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 + 2M \|\nabla u\|_2^2 &\leq C_{\varepsilon_{12}} \|n\|_{\frac{5}{3}}^2 + \varepsilon_{12} \|\nabla u\|_2^2 \\ &\leq C_{\varepsilon_{12}} \|n\|_1^{2\theta} \|n\|_{3(1+\alpha)}^{2(1-\theta)} + \varepsilon_{12} \|\nabla u\|_2^2 \leq C_{\varepsilon_{12}} C_{\varepsilon_{13}} + \varepsilon_{13} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \varepsilon_{13} \|\nabla u\|_2^2, \end{aligned}$$

where $\alpha > 0$. Hence we have

$$(5.29) \quad \frac{M}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 + (2M - \varepsilon_{12}) \|\nabla u\|_2^2 \leq C_{\varepsilon_{12}} C_{\varepsilon_{13}} + \varepsilon_{13} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2.$$

$$(5.30) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \langle x \rangle n \leq C \left(1 + \|\nabla c\|_2^2 + \|\nabla u\|_2^2 \right) + \varepsilon \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \varepsilon \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^2.$$

Summing up (5.26)–(5.30), we have we have for sufficiently small ε_i , $i = 1, \dots, 13$

$$\begin{aligned} (5.31) \quad &\sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n (\log n + 2\langle x \rangle) + \int_{\mathbb{R}^3} n^{\alpha-2q+3} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\ &+ C \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} n |\nabla c|^2 \right) \\ &\leq C \left(1 + \int_0^T \left(\|\nabla c\|_2^2 + 1 \right) \right) \leq C, \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \int_{\mathbb{R}^3} n_0 \log n_0, \|n_0\|_{\alpha-2q+3}, \|n_0(1 + \langle x \rangle)\|_1, \|u_0\|_2 \right)$. Combining estimates (5.10) and (5.31)

$$\begin{aligned} (5.32) \quad &\sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}^3} n |\log n| + \langle x \rangle n + \int_{\mathbb{R}^3} n^{\alpha-q+2} + \int_{\mathbb{R}^3} |\nabla c|^2 + \int_{\mathbb{R}^3} |u|^2 \right) \\ &+ \int_0^T \left(\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 + \|\nabla^2 c\|_2^2 + M \|\nabla u\|_2^2 + \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 \right) \\ &\leq C \left(1 + \int_0^T \left(\|\nabla c\|_2^2 + 1 \right) \right) \leq C, \end{aligned}$$

where $C = C \left(T, \|c_0\|_{L^\infty \cap H^1}, \|n_0(1 + |x| + |\log n_0|)\|_1, \|n_0\|_{\alpha-2q+3}, \|u_0\|_2 \right)$. This completes the proof. \square

6. PROOFS OF THEOREM 1.9 AND THEOREM 1.10

Lemma 6.1. *Let u be a solution of $(1.1)_3$ constructed in Theorem 1.7. If $\alpha > \max\{2q-2, \frac{9q-8}{6}\}$, then $u \in L_t^\infty L_x^6$. Similarly, Suppose that u is a solution of $(1.1)_3$ constructed in Theorem 1.8. If $\alpha > \max\{\min\{2q-2, \frac{9q-8}{6}\}, \frac{10q-9}{8}\}$, then $u \in L_t^\infty L_x^6$.*

Proof. • (Case : $\alpha > \max\{2q-2, \frac{9q-8}{6}\}$).

Since u is a solution of $(1.1)_3$ constructed in Theorem 1.7, we remind that $\int_0^T \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^2 < \infty$. We consider the vorticity equation of $(1.1)_3$

$$(6.1) \quad \omega_t - \Delta \omega = -\nabla \times (n \nabla \phi),$$

where $\omega = \nabla \times u$. The energy estimate yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\omega|^2 + \|\nabla \omega\|_2^2 \leq C \int_{\mathbb{R}^3} |\nabla \omega| \cdot n \leq \frac{1}{2} \|\nabla \omega\|_2^2 + C \|n\|_2^2.$$

Therefore, integrating time, we obtain

$$(6.2) \quad \int_{\mathbb{R}^3} |\omega|^2 + \int_0^T \|\nabla \omega\|_2^2 \leq C \int_0^T \|n\|_2^2.$$

Applying Hölder, interpolation inequalities and Sobolev embedding, we note

$$(6.3) \quad \begin{aligned} \int_0^T \|n\|_2^2 &\leq \int_0^T \|n\|_1^{2\theta} \|n\|_{3(2\alpha-q+2)}^{2(1-\theta)} \leq \int_0^T \|n\|_1^{2\theta} \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-q+2}} \\ &\leq C \int_0^T \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-q+2}}, \end{aligned}$$

where $\theta = (6\alpha - 3q + 4)/(12\alpha - 6q + 10)$. Since $\alpha > \frac{9q-8}{6}$, we note that $4(1-\theta)/(2\alpha-q+2) < 2$, which implies that the righthand side of (6.3) is finite. Therefore, $\omega \in L_{x,t}^{2,\infty}$, which immediately yields $u \in L_{x,t}^{6,\infty}$.

• (Case : $\alpha > \max \{ \min \{ 2q-2, \frac{9q-8}{6} \}, \frac{10q-9}{8} \}$). Since $\max \{ \min \{ 2q-2, \frac{9q-8}{6} \}, \frac{10q-9}{8} \} \geq \frac{10q-9}{8}$, it is enough to consider the case $\frac{10q-9}{8} < \alpha$. We first treat the case that $\frac{10q-9}{8} < \alpha < 2q-1$. We note, due to Hölder, interpolation inequalities and Sobolev embedding, that

$$(6.4) \quad \int_0^T \|n\|_2^2 \leq \int_0^T \|n\|_{\alpha-2q+3}^{2\theta} \|n\|_{3(2\alpha-2q+3)}^{2(1-\theta)} \leq C \int_0^T \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-2q+3}},$$

where

$$\theta = \frac{(\alpha - 2q + 3)(6\alpha - 6q - 7)}{2(5\alpha - 4q + 6)}$$

and we used that $\|n\|_{L_t^\infty L_x^{\alpha-2q+3}} < C$ in Theorem 1.8. Observing that $4(1-\theta)/2\alpha-2q+3 < 2$, we can see that the righthand side of (6.4) is finite. For the case that $\alpha \geq 2q-1$, using a different interpolation inequality, we estimate

$$(6.5) \quad \int_0^T \|n\|_2^2 \leq \int_0^T \|n\|_1^{2\theta} \|n\|_{3(2\alpha-2q+3)}^{2(1-\theta)} \leq C \int_0^T \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^{2(1-\theta) \frac{2}{2\alpha-2q+3}},$$

where

$$\theta = \frac{6\alpha - 6q + 7}{2(6\alpha - 6q + 8)}.$$

Since $\alpha \geq 2q-1$, we note that $4(1-\theta)/2\alpha-2q+3 < 2$, which implies that the righthand side of (6.5) is finite. Due to estimates (6.3), (6.4) and (6.5), we deduce the lemma. \square

We present proof of Theorem 1.9.

Proof of Theorem 1.9. • (Case $\max \{ 2q-2, \frac{9q-8}{6} \} < \alpha$ and $\frac{3q-1}{6} \leq \alpha$)
Multiplying equation (1.1)₁ with n^{p-1} and using the integration by parts, we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |n|^p + \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 &\leq \frac{p-1}{p+q-1} \int_{\mathbb{R}^3} \chi(c) |\nabla c| |\nabla n^{p+q-1}| \\ &\leq \frac{p-1}{p+q-1} \int_{\mathbb{R}^3} \chi(c) |\nabla c| |\nabla n^{\frac{p+\alpha}{2}}| n^{\frac{p+2q-\alpha-2}{2}}. \end{aligned}$$

Applying Hölder and Young's inequalities, we have

$$(6.6) \quad \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |n|^p + \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \leq \varepsilon_1 \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 + C_{\varepsilon_1} \int_{\mathbb{R}^3} n^{p+2q-\alpha-2} |\nabla c|^2.$$

The right hand side of the above is estimated as

$$\begin{aligned} \int_{\mathbb{R}^3} n^{p+2q-\alpha-2} |\nabla c|^2 &\leq \left\| n^{p+2q-\alpha-2} \right\|_{\frac{p}{p+2q-\alpha-2}} \left\| |\nabla c|^2 \right\|_{\frac{p}{\alpha+2-2q}} \\ &\leq \left\| n \right\|_p^{p+2q-\alpha-2} \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \leq C \left(1 + \left\| n \right\|_p^p \right) \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2. \end{aligned}$$

Therefore, taking $\varepsilon_1 \leq \frac{2(p-1)(1+\alpha)}{(p+\alpha)^2}$, we obtain

$$\frac{d}{dt} \left\| n \right\|_p^p \leq Cp^2 \left(1 + \left\| n \right\|_p^p \right) \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2.$$

Gronwall inequality implies that

$$(6.7) \quad \sup_{0 \leq t \leq T} \left\| n \right\|_p^p \leq \exp \left\{ Cp^2 \int_0^T \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right\} \int_0^T \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \left\| n_0 \right\|_p^p.$$

We will show that $\int_0^T \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 < \infty$ for any $p > \alpha - q + 2$. Indeed, from maximal regularity theorem for heat equation, we have

$$(6.8) \quad \begin{aligned} \int_0^T \left\| \nabla^2 c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 &\leq C \int_0^T \left(\left\| n \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \left\| u \cdot \nabla c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + C \left\| \nabla^2 c_0 \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \\ &= C(\text{I} + \text{II}) + C \left\| \nabla^2 c_0 \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2. \end{aligned}$$

The term I is estimated as follows. Since $\frac{3q-1}{6} \leq \alpha$, we have via interpolation inequality and Sobolev embedding

$$(6.9) \quad \text{I} = \int_0^T \left\| n \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \leq \int_0^T \left\| n \right\|_1^{2\theta} \left\| n \right\|_{\frac{6p}{3(2\alpha-q+2)}}^{2(1-\theta)} \leq \int_0^T \left\| \nabla n^{\frac{2\alpha-q+2}{2}} \right\|_2^{2-\delta_p},$$

where

$$\begin{aligned} 2(1-\theta) &= \frac{(2\alpha-q+2)(4p-3\alpha-6+6q)}{p(6\alpha-3q+5)} \quad \text{and} \\ 2(1-\theta) \frac{2}{2\alpha-q+2} &= \frac{8}{6\alpha-3q+5} - \frac{6\alpha+12-12q}{p(6\alpha-3q+5)} = 2 - \delta_p < 2. \end{aligned}$$

Thus, it is direct that the term I is finite. On the other hand, applying Hölder inequality and Lemma 6.1, we estimate the term II as follows.

$$\text{II} = \int_0^T \left\| u \cdot \nabla c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \leq \int_0^T \left\| u \right\|_6^2 \left\| \nabla c \right\|_{\frac{6p}{p+3\alpha+6-6q}}^2 \leq C \int_0^T \left\| \nabla^2 c \right\|_{\frac{2p}{p+\alpha+2-2q}}^2.$$

Using the maximal regularity for heat equation, interpolation inequality, Sobolev embedding and $\frac{2p}{p+\alpha+2-2q} < \frac{6p}{2p+3\alpha+6-6q}$, we have

$$\begin{aligned} \int_0^T \left\| \nabla^2 c \right\|_{\frac{2p}{p+\alpha+2-2q}}^2 &\leq C \int_0^T \left(\left\| n \right\|_{\frac{2p}{p+\alpha+2-2q}}^2 + \left\| u \cdot \nabla c \right\|_{\frac{2p}{p+\alpha+2-2q}}^2 \right) + \left\| \nabla^2 c_0 \right\|_{\frac{2p}{p+\alpha+2-2q}}^2 \\ &\leq C \int_0^T \left(\left\| n \right\|_1^2 + \left\| n \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \left\| u \right\|_6^2 \left\| \nabla c \right\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + \left\| \nabla^2 c_0 \right\|_{\frac{2p}{p+\alpha+2-2q}}^2. \end{aligned}$$

$$(6.10) \quad \leq C \int_0^T \left(\|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|\nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + CT + \|\nabla^2 c_0\|_{\frac{2p}{p+\alpha+2-2q}}^2.$$

We note that the first term in (6.10) is the same as I in (6.9) and thus it is finite. It is straightforward that $\int_0^T \|\nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 < \infty$, since $2 < \frac{6p}{2p+3\alpha+6-6q} < 3$. Hence, the second term II is also finite, which deduces the boundedness of L^p -norm of n , for any $p > \alpha - q + 2$.

• (Case $\max\{\frac{9q-8}{6}, 2q-2\} < \alpha < \frac{3q-1}{6}$) We first note that this case is equivalent to the case that $\frac{9q-8}{6} < \alpha < \frac{3q-1}{6}$ with $1 \leq q < \frac{7}{6}$. We set p with

$$(6.11) \quad \max\{\alpha - q + 2, 3\alpha - 4q + 4\} < p < 4\alpha - 5q + 6.$$

Testing n^{p-1} to equation (1.1)₁ and following similar computations as in previous case, we have

$$(6.12) \quad \frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^3} |n|^p + \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \leq \varepsilon_2 \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 + C_{\varepsilon_2} \int_{\mathbb{R}^3} n^{p+2q-\alpha-2} |\nabla c|^2.$$

Noting that

$$\begin{aligned} \int_{\mathbb{R}^3} n^{p+2q-\alpha-2} |\nabla c|^2 &= \int_{\mathbb{R}^3} n^{p+2q-\alpha-2} \nabla c \cdot \nabla c \\ &\leq \int_{\mathbb{R}^3} |\nabla n^{p+2q-\alpha-2}| c |\nabla c| + \int_{\mathbb{R}^3} |n^{p+2q-\alpha-2}| c |\nabla^2 c| \end{aligned}$$

and integrating in time, we estimate (6.12) as

$$(6.13) \quad \sup_{0 \leq t \leq T} \|n\|_p^p + \left(\frac{4p(p-1)(1+\alpha)}{(p+\alpha)^2} - \varepsilon_2 \right) \int_0^T \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2$$

$$\leq Cp^2 \int_0^T \int_{\mathbb{R}^3} |\nabla n^{p+2q-\alpha-2}| |\nabla c| + Cp^2 \int_0^T \int_{\mathbb{R}^3} |n^{p+2q-\alpha-2}| |\nabla^2 c| := Cp^2 (\text{I} + \text{II}).$$

We first estimate I. Applying Hölder and Young's inequalities, we observe that

$$(6.14) \quad \text{I} \leq \int_0^T \int_{\mathbb{R}^3} |\nabla n^{\frac{p+\alpha}{2}}| n^{\frac{p+4q-3\alpha-4}{2}} |\nabla c| \leq \int_0^T \left(\varepsilon_3 \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 + C_{\varepsilon_3} \int_{\mathbb{R}^3} n^{p+4q-3\alpha-4} |\nabla c|^2 \right).$$

Let $r_1 = \frac{6(\alpha-q+2)}{14\alpha-17q+22-3p}$. Due to (6.11), Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} n^{p+4q-3\alpha-4} |\nabla c|^2 \\ &\leq \int_0^T \left\| n^{p+4q-3\alpha-4} \right\|_{\frac{\alpha-q+2}{p+4q-3\alpha-4}} \left\| |\nabla c|^2 \right\|_{\frac{\alpha-q+2}{4\alpha-5q+6-p}} \\ &\leq \int_0^T \|n\|_{\alpha-q+2}^{p+4q-3\alpha-4} \|\nabla^2 c\|_{r_1}^2 \leq C \int_0^T \|\nabla^2 c\|_{r_1}^2, \end{aligned}$$

where we used that $\|n\|_{L_t^\infty L_x^{\alpha-q+2}} < C$ proved in Theorem 1.7. From maximal regularity for heat equation and results in Lemma 6.1, we note

$$\int_0^T \|\nabla^2 c\|_{r_1}^2 \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|u \cdot \nabla c\|_{r_1}^2 \right) + C \|\nabla^2 c_0\|_{r_1}^2$$

$$\begin{aligned}
&\leq C \int_0^T \left(\|n\|_{r_1}^2 + \|u\|_6^2 \|\nabla c\|_{\frac{6r_1}{6-r_1}}^2 \right) + C \|\nabla^2 c_0\|_{r_1}^2 \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|\nabla^2 c\|_{\frac{6r_1}{6+r_1}}^2 \right) + C \|\nabla^2 c_0\|_{r_1}^2 \\
&\leq C \int_0^T \left(\|n\|_{r_1}^2 + \|n\|_{\frac{6r_1}{6+r_1}}^2 + \|u\|_6^2 \|\nabla c\|_{r_1}^2 \right) + C \left(\|\nabla^2 c_0\|_{r_1}^2 + \|\nabla^2 c_0\|_{\frac{6r_1}{6+r_1}}^2 \right) \\
(6.15) \quad &\leq C \int_0^T \left(\|n\|_{r_1}^2 + \|n\|_1^2 + \|u\|_6^2 \|\nabla c\|_{r_1}^2 \right) + C := I_1
\end{aligned}$$

If $r_1 \leq \alpha - q + 2$, then (6.15) is bounded, due to the result of Theorem 1.7, by

$$(6.16) \quad I_1 \leq C \int_0^T \|\nabla c\|_{r_1}^2 + C(1+T).$$

On the other hand, in case that $r_1 > \alpha - q + 2$, we can see that $r_1 < 3(p + \alpha)$, since $\max\{\alpha - q + 2, 3\alpha - 4q + 4\} < p < 4\alpha - 5q + 6$, and thus (6.15) is estimated as

$$\begin{aligned}
I_1 &\leq C \int_0^T \left(\|n\|_{\alpha-q+2}^{2\theta_1} \|n\|_{3(p+\alpha)}^{2(1-\theta_1)} + \|\nabla c\|_{r_1}^2 \right) + C(1+T) \\
(6.17) \quad &\leq C \int_0^T \left(\left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^{\delta_1} + \|\nabla c\|_{r_1}^2 \right) + C(1+T),
\end{aligned}$$

where

$$2(1 - \theta_1) = \frac{6(p + \alpha)(r_1 - (\alpha - q + 2))}{r_1(3p + 2\alpha + q - 2)}, \quad \delta_1 = \frac{12(r_1 - \alpha + q - 2)}{r_1(3p + 2\alpha + q - 2)}.$$

Here we used that $\alpha - q + 2 < r_1 < 3(p + \alpha)$ and $r_1 \geq \frac{6r_1}{r_1+6}$. We note that $\delta_1 < 2$, since $\max\{2q - 2, \frac{9q-8}{6}\} < \alpha < \frac{3q-1}{6}$. Next we estimate the term II. Let $r_2 := p + 2q - \alpha - 1$. Using Hölder, Young's and maximal regularity for heat equation, and following similar computations as in (6.15), II is estimated as follows:

$$\begin{aligned}
II &\leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|\nabla^2 c\|_{r_2}^{r_2} \right) \leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|u \cdot \nabla c\|_{r_2}^{r_2} + C \|\nabla^2 c_0\|_{r_2}^{r_2} \right) \\
&\leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|n\|_{\frac{6r_2}{6+r_2}}^{r_2} + \|\nabla c\|_{r_2}^{r_2} \right) + C \left(\|\nabla^2 c_0\|_{r_2}^{r_2} + \|\nabla^2 c_0\|_{\frac{6r_1}{6+r_1}}^2 \right).
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_0^T \left(\|n\|_{r_2}^{r_2} + \|n\|_{\frac{6r_2}{6+r_2}}^{r_2} + \|\nabla c\|_{r_2}^{r_2} \right) \leq \int_0^T \left(C\|n\|_{r_2}^{r_2} + C\|n\|_1^{r_2} + \|\nabla c\|_{r_2}^{r_2} \right) \\
&\leq C \int_0^T \left(\|n\|_{\alpha-q+2}^{r_2\theta_3} \|n\|_{3(p+\alpha)}^{r_2(1-\theta_3)} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla c\|_6^{r_2(1-\theta_4)} \right) + CT \\
&\leq C \int_0^T \left(\|n\|_{\alpha-q+2}^{r_2\theta_3} \|n\|_{3(p+\alpha)}^{r_2(1-\theta_3)} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla^2 c\|_2^{r_2(1-\theta_4)} \right) + CT \\
&\leq C \int_0^T \left(\left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^{\delta_3} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla^2 c\|_2^{\delta_4} \right),
\end{aligned}$$

where

$$\delta_3 = \frac{2r_2(1 - \theta_3)}{p + \alpha} = \frac{6(p - 2\alpha + 3q - 3)}{3p + 2\alpha + q - 2}, \quad \delta_4 = r_2(1 - \theta_4) = \frac{3}{2}(p + 2q - \alpha - 3).$$

Here we used that $\alpha - q + 2 < r_2 < 3(p + \alpha)$ and $2 < r_2 < 6$ and we observe that $\delta_3 < 2$ and $\delta_4 < 2$, since $\frac{9q-8}{6} < \alpha < \frac{3q-1}{6}$ and $\max\{\alpha - q + 2, 3\alpha - 4q + 4\} < p < 4\alpha - 5q + 6$.

Combining estimates of I and II, we obtain

$$\sup_{0 \leq t \leq T} \|n(t)\|_p^p + C \int_0^T \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \leq C \left(\int_0^T \|\nabla c\|_{r_1}^2 + 1 \right).$$

We can see also that $2 < r_1 < 6$ as long as $\frac{11\alpha-14q+16}{3} < p < 4\alpha - 5q + 6$, which is valid, since $\frac{11\alpha-14q+16}{3} > 3\alpha - 4q + 4$, in case that $\frac{9q-8}{6} < \alpha < \frac{3q-1}{6}$. Therefore, for any p with $\max\{\alpha - q + 2, 3\alpha - 4q + 4\} < p < 4\alpha - 5q + 6$ we obtain

$$(6.18) \quad n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3))$$

Let $p_0 = \frac{6-3\alpha}{4}$. We then see that $1 < p_0 < 4\alpha - 5q + 6$ via $\frac{9q-8}{6} < \alpha < \frac{3q-1}{6}$, and thus it is evident from (6.18) that

$$(6.19) \quad n \in L^\infty(0, T; L^{p_0}(\mathbb{R}^3)).$$

Next, we will show that $n \in L^\infty(0, T; L^p(\mathbb{R}^3))$ for any $p_0 < p < \infty$. Similarly as before, multiplying equation (1.1)₁ with n^{p-1} and using Gronwall inequality, we get (6.7), namely,

$$\sup_{0 \leq t \leq T} \|n\|_p^p \leq \exp \left\{ Cp^2 \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right\} \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|n_0\|_p^p.$$

We recall (6.8) via maximal regularity for heat equation, i.e.

$$\begin{aligned} \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 &\leq C \int_0^T \left(\|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|u \cdot \nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + C \|\nabla^2 c_0\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \\ &= C(\text{III} + \text{IV}) + C \|\nabla^2 c_0\|_{\frac{6p}{2p+3\alpha+6-6q}}^2. \end{aligned}$$

The term III is estimated as follows. For $p > \alpha - q + 2$, we have $p_0 < \frac{6p}{2p+3\alpha+6-6q} < 3p_0 + 3\alpha$ and hence it follows from (6.19) that

$$(6.20) \quad \text{III} = \|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \leq \|n\|_{p_0}^{2\theta} \|n\|_{3(p_0+\alpha)}^{2(1-\theta)} \leq \|n\|_{p_0}^{2\theta} \left\| \nabla n^{\frac{p_0+\alpha}{2}} \right\|_2^{2-\delta_p},$$

where

$$2(1-\theta) \frac{p_0 + \alpha}{2} = \frac{12 - 4p_0}{2p_0 + 3\alpha} - \frac{2p_0(3\alpha + 6 - 6q)}{p(2p_0 + 3\alpha)} = 2 - \delta_p.$$

The term IV is estimated exactly in the same way as II in, and thus we obtain

$$(6.21) \quad \text{IV} \leq C \int_0^T \left(\|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|\nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + CT + \|\nabla^2 c_0\|_{\frac{2p}{p+\alpha+2-2q}}^2.$$

The first term in (6.21) can be treated as the case III and the second term is bounded, since $2 < \frac{6p}{2p+3\alpha+6-6q} < 3$ and $\int_0^T \|\nabla c\|_m^2 ds < \infty$ for $2 \leq m \leq 3$. We finally conclude the boundedness of L^∞ -norm of n . Indeed, since $n \in L^\infty(0, T; L^p(\mathbb{R}^3))$ for all $1 \leq p < \infty$, we can see that c_t , ∇c , u_t and $\nabla^2 u$ belong to $L^p((0, T) \times \mathbb{R}^3)$ for all $p < \infty$ and therefore, we also note, due to parabolic embedding, that $\nabla c \in L^\infty((0, T) \times \mathbb{R}^3)$. Using estimate (6.6) and $\nabla c \in L^\infty((0, T) \times \mathbb{R}^3)$, we obtain

$$(6.22) \quad \frac{d}{dt} \int_{\mathbb{R}^3} |n|^p \leq Cp^2 \int_{\mathbb{R}^3} n^{p(1-\delta)}, \quad \delta := \frac{\alpha + 2 - 2q}{p}.$$

Multiplying equation (6.22) with p , we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |n|^p dx \leq Cp^2 \|n\|_{p(1-\delta)}^{p(1-\delta)}.$$

Using interpolation inequality, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |n|^p dx \leq Cp^2 \|n\|_p^{p(1-\delta)(1-\theta)},$$

where $p(1-\delta)(1-\theta) = \frac{p(p(1-\delta)-1)}{p-1}$. Let $y(t) := \|n(t)\|_p^p$ and $\beta := 1 - p(1-\delta)(1-\theta)$. Then we have

$$y(t)' \leq Cp^2 y(t)^{1-\beta}.$$

Via Gronwall inequality, we observe that

$$(6.23) \quad \|n(t)\|_p \leq (Cp^2 \beta t)^{\frac{1}{\beta p}} + \|n(0)\|_p, \quad t \leq T.$$

Passing p to the limit, we obtain for all $t \leq T$.

$$\|n(t)\|_\infty \leq 1 + \|n(0)\|_\infty.$$

Hölder continuity is a direct consequence of Theorem 1.2. Indeed, since $n \in L_t^\infty(L_x^1 \cap L_x^\infty)$, due to Lemma 2.7 and Lemma 2.8, we obtain

$$v_t, \nabla^2 v, c_t, \nabla^2 c \in L_{x,t}^l \quad \text{for all } 1 < l < \infty.$$

In our case, $B = u + \chi(c)\nabla c$ and we note, due to parabolic embedding, that B satisfies the condition (1.3). Therefore, we conclude that $n \in \mathcal{C}_{x,t}^\beta$ for some $\beta > 0$.

Due to classical Schauder estimates, u and c are also in the class $\mathcal{C}_{x,t}^{2+\beta, 1+\frac{\beta}{2}}$. This completes the proof. \square

We present proof of Theorem 1.10.

Proof of Theorem 1.10.

- (Case : $\max\{\min\{2q-2, \frac{9q-8}{6}\}, \frac{10q-9}{8}\} < \alpha$ and $\frac{3q-1}{6} \leq \alpha$)

We note first that this case is reduced to the case that $\min\{2q-2, \frac{9q-8}{6}\} < \alpha$, $\frac{3q-1}{6} \leq \alpha$ and $q \geq \frac{7}{6}$. We also observe that, in the case that $\alpha \geq 2q$, the result is already obtained in Theorem 1.9, and therefore, it suffices to treat the case $\alpha < 2q$, that is $\min\{2q-2, \frac{9q-8}{6}\} < \alpha$, $\frac{3q-1}{6} \leq \alpha < 2q$ and $q \geq \frac{7}{6}$.

As in the previous case (6.8), following similar computations, we have

$$(6.24) \quad \sup_{0 \leq t \leq T} \|n\|_p^p \leq \exp \left\{ Cp^2 \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right\} \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|n_0\|_p^p.$$

We will show that $\int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 < \infty$ for any $p > \alpha - 2q + 3$. Indeed, we recall (6.8) via maximal regularity for heat equation

$$(6.25) \quad \begin{aligned} \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 &\leq C \int_0^T \left(\|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|u \cdot \nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + C \|\nabla^2 c_0\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \\ &= C(\text{I} + \text{II}) + C \|\nabla^2 c_0\|_{\frac{6p}{2p+3\alpha+6-6q}}^2. \end{aligned}$$

The term I is estimated as follows. Since $\min \{2q-2, \frac{9q-8}{6}\} < \alpha$ and $\frac{3q-1}{6} \leq \alpha < 2q$, we have via interpolation inequality and Sobolev embedding

$$(6.26) \quad \text{I} = \int_0^T \|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \leq \int_0^T \|n\|_{\alpha-2q+3}^{2\theta} \|n\|_{3(2\alpha-2q+3)}^{2(1-\theta)} \leq \int_0^T \left\| \nabla n^{\frac{2\alpha-2q+3}{2}} \right\|_2^{2-\delta_p},$$

where

$$2(1-\theta) = \frac{(2\alpha-2q+3)\{p(-2\alpha+4q) - (\alpha-2q+3)(2p+3\alpha+6-6q)\}}{p(5\alpha-4q+6)} \quad \text{and}$$

$$2(1-\theta)\frac{2}{2\alpha-2q+3} = \frac{4q-2\alpha}{5\alpha-4q+6} - \frac{(\alpha-2q+3)(2p+3\alpha+6-6q)}{p(5\alpha-4q+6)} = 2-\delta_p < 2.$$

Thus, it is direct that the term I is finite.

On the other hand, the second term II can be computed exactly as the same way as that of (6.10) in Theorem 1.9, and thus the details are omitted.

- (Case : $\max \{ \min \{2q-2, \frac{9q-8}{6}\}, \frac{10q-9}{8} \} < \alpha < \frac{3q-1}{6}$)

We note that this is reduced to the case that $\frac{10q-9}{8} < \alpha < \frac{3q-1}{6}$ and $1 \leq q < \frac{7}{6}$. We set p with

$$(6.27) \quad \max \{ \alpha - 2q + 3, 3\alpha - 4q + 4 \} < p < 4\alpha - 6q + 7.$$

Similarly as before, testing n^{p-1} to $(1.1)_1$, we obtain (6.13), namely

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|n\|_p^p + \int_0^T \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \\ & \leq Cp^2 \int_0^T \int_{\mathbb{R}^3} |\nabla n^{p+2q-\alpha-2}| |\nabla c| + Cp^2 \int_0^T \int_{\mathbb{R}^3} |n^{p+2q-\alpha-2}| |\nabla^2 c| := Cp^2 (\text{I} + \text{II}). \end{aligned}$$

The first term I is the same as (6.14).

$$\text{I} \leq \int_0^T \left(\varepsilon \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 + C_\varepsilon \int_{\mathbb{R}^3} n^{p+4q-3\alpha-4} |\nabla c|^2 \right).$$

Let $r_1 = \frac{6(\alpha-2q+3)}{14\alpha-22q+27-3p}$. Due to (6.27), Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} n^{p+4q-3\alpha-4} |\nabla c|^2 & \leq \int_0^T \|n^{p+4q-3\alpha-4}\|_{\frac{\alpha-2q+3}{p+4q-3\alpha-4}} \|\nabla c\|_{\frac{\alpha-2q+3}{4\alpha-6q+7-p}}^2 \\ & \leq \int_0^T \|n\|_{\alpha-2q+3}^{p+4q-3\alpha-4} \|\nabla^2 c\|_{r_1}^2 \leq C \int_0^T \|\nabla^2 c\|_{r_1}^2, \end{aligned}$$

where we used that $\|n\|_{L_t^\infty L_x^{\alpha-2q+3}} < C$ proved in Theorem 1.8. From maximal regularity for heat equation and results in Lemma 6.1, we have

$$\begin{aligned} \int_0^T \|\nabla^2 c\|_{r_1}^2 & \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|u \cdot \nabla c\|_{r_1}^2 \right) + \|\nabla^2 c_0\|_{r_1}^2 \\ & \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|u\|_6^2 \|\nabla c\|_{\frac{6r_1}{6-r_1}}^2 \right) + \|\nabla^2 c_0\|_{r_1}^2 \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|\nabla^2 c\|_{\frac{6r_1}{6-r_1}}^2 \right) + \|\nabla^2 c_0\|_{r_1}^2 \\ & \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|n\|_{\frac{6r_1}{6+r_1}}^2 + \|u\|_6^2 \|\nabla c\|_{r_1}^2 \right) + C \left(\|\nabla^2 c_0\|_{r_1}^2 + \|\nabla^2 c_0\|_{\frac{6r_1}{6+r_1}}^2 \right) \end{aligned}$$

$$(6.28) \quad \leq C \int_0^T \left(\|n\|_{r_1}^2 + \|n\|_1^2 + \|u\|_6^2 \|\nabla c\|_{r_1}^2 \right) + C := I_1$$

If $r_1 \leq \alpha - 2q + 3$, then (6.28) is bounded, due to the result of Theorem 1.8, by

$$(6.29) \quad I_1 \leq C \int_0^T \|\nabla c\|_{r_1}^2 + C(1+T).$$

On the other hand, in case that $r_1 > \alpha - 2q + 3$, we can see that $r_1 < 3(p+\alpha)$, since $\max\{\alpha - 2q + 3, 3\alpha - 4q + 4\} < p < 4\alpha - 6q + 7$, and thus (6.28) is estimated as

$$(6.30) \quad \begin{aligned} I_1 &\leq C \int_0^T \left(\|n\|_{\alpha-2q+3}^{2\theta_1} \|n\|_{3(p+\alpha)}^{2(1-\theta_1)} + \|\nabla c\|_{r_1}^2 \right) + C(1+T) \\ &\leq C \int_0^T \left(\left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^{\delta_1} + \|\nabla c\|_{r_1}^2 \right) + C(1+T), \end{aligned}$$

where

$$2(1-\theta_1) = \frac{(p+\alpha)(3p-14\alpha+22q-21)}{3p+2\alpha+2q-3}, \quad \delta_1 = \frac{2(3p-14\alpha+22q-21)}{3p+2\alpha+2q-3}.$$

Here we used that $\alpha - 2q + 3 < r_1 < 3(p+\alpha)$ and $r_1 \geq \frac{6r_1}{r_1+6}$. We note that $\delta_1 < 2$, since $\frac{10q-9}{8} < \alpha < \frac{3q-1}{6}$.

Next we estimate the term II. Let $r_2 := p + 2q - \alpha - 1$. Using Hölder, Young's and maximal regularity for heat equation, and following similar computations as in (6.28), II is estimated as follows:

$$\begin{aligned} \text{II} &\leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|\nabla^2 c\|_{r_2}^{r_2} \right) \leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|u \cdot \nabla c\|_{r_2}^{r_2} \right) + C \|\nabla^2 c_0\|_{r_2}^{r_2} \\ &\leq C \int_0^T \left(\|n\|_{r_2}^{r_2} + \|n\|_{\frac{6r_2}{6+r_2}}^{\frac{r_2}{6+r_2}} + \|\nabla c\|_{r_2}^{r_2} \right) + C \left(\|\nabla^2 c_0\|_{r_2}^{r_2} + \|\nabla^2 c_0\|_{\frac{6r_1}{6+r_1}}^2 \right). \end{aligned}$$

We note that

$$\begin{aligned} \int_0^T \left(\|n\|_{r_2}^{r_2} + \|n\|_{\frac{6r_2}{6+r_2}}^{\frac{r_2}{6+r_2}} + \|\nabla c\|_{r_2}^{r_2} \right) &\leq \int_0^T \left(C\|n\|_{r_2}^{r_2} + C\|n\|_1^{r_2} + \|\nabla c\|_{r_2}^{r_2} \right) \\ &\leq C \int_0^T \left(\|n\|_{\alpha-2q+3}^{r_2\theta_3} \|n\|_{3(p+\alpha)}^{r_2(1-\theta_3)} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla c\|_6^{r_2(1-\theta_4)} \right) + CT \\ &\leq C \int_0^T \left(\|n\|_{\alpha-2q+3}^{r_2\theta_3} \|n\|_{3(p+\alpha)}^{r_2(1-\theta_3)} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla^2 c\|_2^{r_2(1-\theta_4)} \right) + CT \\ &\leq C \int_0^T \left(\left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^{\delta_3} + \|\nabla c\|_2^{r_2\theta_4} \|\nabla^2 c\|_2^{\delta_4} \right), \end{aligned}$$

where

$$\delta_3 = \frac{2r_2(1-\theta_3)}{p+\alpha} = \frac{6(p-2\alpha+4q-4)}{3p+2\alpha+2q-4}, \quad \delta_4 = r_2(1-\theta_4) = \frac{3}{2}(p+2q-\alpha-3).$$

Here we used that $\alpha - 2q + 3 < r_2 < 3(p+\alpha)$ and $2 < r_2 < 6$ and we observe that $\delta_3 < 2$ and $\delta_4 < 2$, since $\frac{10q-9}{8} < \alpha < \frac{3q-1}{6}$ and $\max\{\alpha - 2q + 3, 3\alpha - 4q + 4\} < p < 4\alpha - 6q + 7$. Combining estimates of I and II, we obtain

$$\sup_{0 \leq t \leq T} \|n(t)\|_p^p + C \int_0^T \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 \leq C \left(\int_0^T \|\nabla c\|_{r_1}^2 + 1 \right).$$

We can see also that $2 < r_1 < 6$ as long as $\frac{11\alpha-16q+18}{3} < p < 4\alpha - 6q + 7$, which is valid, since $\frac{11\alpha-16q+18}{3} > 3\alpha - 4q + 4$, in case that $\frac{10q-9}{8} < \alpha < \frac{3q-1}{6}$. Therefore, for any p with $\max\{\alpha - 2q + 3, 3\alpha - 4q + 4\} < p < 4\alpha - 6q + 7$ we obtain

$$(6.31) \quad n \in L^\infty(0, T; L^p(\mathbb{R}^3)), \quad \nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^3))$$

Let $p_0 = \frac{6-3\alpha}{4}$. We can see that $1 < p_0 < 4\alpha - 6q + 7$ via $\frac{10q-9}{8} < \alpha < \frac{3q-1}{3}$, and thus it is evident from (6.31) that

$$(6.32) \quad n \in L^\infty(0, T; L^{p_0}(\mathbb{R}^3)).$$

Next, we will show that $n \in L^\infty(0, T; L^p(\mathbb{R}^3))$ for any $p_0 < p < \infty$. Similarly as before, multiplying equation (1.1)₁ with n^{p-1} and using Gronwall inequality, we get (6.7), namely,

$$\sup_{0 \leq t \leq T} \|n\|_p^p \leq \exp \left\{ Cp^2 \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right\} \int_0^T \|\nabla^2 c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|n_0\|_p^p.$$

From maximal regularity for heat equation, we have

$$\begin{aligned} \int_0^T \|\Delta c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 &\leq C \int_0^T \left(\|n\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 + \|u \cdot \nabla c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \right) + \int_0^T \|\Delta c_0\|_{\frac{6p}{2p+3\alpha+6-6q}}^2 \\ &= C(\text{III} + \text{IV}) + \int_0^T \|\Delta c\|_{\frac{6p}{2p+3\alpha+6-6q}}^2. \end{aligned}$$

For $p > \alpha - 2q + 3$, we have $p_0 < \frac{6p}{2p+3\alpha+6-6q} < 3p_0 + 3\alpha$ and thus, the terms III and IV are estimated as exactly the same as (6.20) and (6.21). Hence, we skip its details. We finally conclude the boundedness of L^∞ -norm of n . Indeed, since $n \in L^\infty(0, T; L^p(\mathbb{R}^3))$ for all $1 \leq p < \infty$, we can see that c_t , ∇c , u_t and $\nabla^2 u$ belong to $L^p((0, T) \times \mathbb{R}^3)$ for all $p < \infty$ and therefore, we also note, due to parabolic embedding, that $\nabla c \in L^\infty((0, T) \times \mathbb{R}^3)$. Following similar procedure as (6.22) in Theorem 1.10, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |n|^p dx \leq Cp^2 \|n\|_{p(1-\delta)}^{p(1-\delta)}.$$

Due to exactly same computations as in (6.23), we obtain

$$\|n(t)\|_\infty \leq 1 + \|n(0)\|_\infty.$$

Hölder continuity can be verified similarly as in Theorem 1.9, and thus the details are skipped. This completes the proof. \square

7. APPENDIX

7.1. Proof of Local energy estimates, Propositions 4.1 and 4.2.

Proof of Proposition 4.1: For a nonnegative bounded weak solution n , set up the test functions

$$\varphi_\pm = \pm 2(n - \mu_\pm \pm k)_\pm \zeta^2,$$

where ζ is a piecewise linear cutoff function vanishing on the parabolic boundary of Q_ρ . We calculate first that

$$\begin{aligned} I &= \iint_{Q_\rho} n_t \varphi_\pm dx dt = \iint_{Q_\rho} \frac{\partial}{\partial t} [(n - \mu_\pm \pm k)_\pm^2] \zeta^2 dx dt \\ &= \int_{K_\rho \times \{t_1\}} (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx - \int_{K_\rho \times \{t_0\}} (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx \\ &\quad - 2 \iint_{Q_\rho} (n - \mu_\pm \pm k)_\pm^2 \zeta \zeta_t dx dt. \end{aligned}$$

Now we consider the following integral quantities:

$$II = \iint_{Q_\rho} \nabla n^{1+\alpha} \nabla \varphi_\pm dx dt = II_1 + II_2$$

where

$$II_1 = 2(1 + \alpha) \iint_{Q_\rho} n^\alpha |\nabla(n - \mu_\pm \pm k)_\pm|^2 \zeta^2 dx dt,$$

and (for any $\epsilon_0 > 0$ by the Cauchy-Schwartz inequality)

$$\begin{aligned} II_2 &= 2(1 + \alpha) \iint_{Q_\rho} n^\alpha \nabla(n - \mu_\pm \pm k)_\pm (n - \mu_\pm \pm k)_\pm \zeta \nabla \zeta dx dt \\ &\leq 2\epsilon_0(1 + \alpha) \iint_{Q_\rho} n^\alpha |\nabla(n - \mu_\pm \pm k)_\pm|^2 \zeta^2 dx dt \\ &\quad + 8\epsilon_0^{-1}(1 + \alpha) \iint_{Q_\rho} n^\alpha (n - \mu_\pm \pm k)_\pm^2 |\nabla \zeta|^2 dx dt = II_{21} + II_{22}. \end{aligned}$$

The first term on the right hand side is absorbed to II_1 by choosing $\epsilon_0 = 1/2$.

Now we consider integral terms carrying the lower order term,

$$III = \iint_{Q_\rho} \nabla(Bn) \varphi_\pm dx dt = III_1 + III_2$$

where

$$III_1 = \iint_{Q_\rho} \nabla B n \varphi_\pm dx dt,$$

and (by taking integration by parts with respect to the space variable)

$$\begin{aligned} III_2 &= \iint_{Q_\rho} B \nabla n \varphi_\pm dx dt = \iint_{Q_\rho} B [\nabla(n - \mu_\pm \pm k)_\pm^2] \zeta^2 dx dt \\ &= - \iint_{Q_\rho} \nabla B (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx dt - 2 \iint_{Q_\rho} B (n - \mu_\pm \pm k)_\pm^2 \zeta \nabla \zeta dx dt \\ &= III_{21} + III_{22}. \end{aligned}$$

Then by applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned} III_{22} &\leq \iint_{Q_\rho} k^\alpha (n - \mu_\pm \pm k)_\pm^2 |\nabla \zeta|^2 dx dt \\ &\quad + \iint_{Q_\rho} |B|^2 k^{-\alpha} (n - \mu_\pm \pm k)_\pm^2 \zeta^2 dx dt = III_{221} + III_{222}, \end{aligned}$$

where the first term, III_{221} , and II_{22} are collected together in (4.2) (the third term on the right-hand-side). Notice that $(n - \mu_{\pm} \pm k)_{\pm} \leq k$. From condition (1.3), we compute

$$III_{222} \leq k^{2-\alpha} \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^{\pm}(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}},$$

$$III_1 + III_{21} \leq (2k\mu_+ + k^2) \|\nabla B\|_{\hat{q}_1, \hat{q}_2} \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^{\pm}(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}.$$

Proof of Proposition 4.2 : Due to the setting of function Ψ_{\pm} in (4.3), we compute that

$$\Psi'_{\pm} = \frac{\pm 1}{(1+\delta)k - (n - \mu_{\pm} \pm k)_{\pm}} \quad \text{and} \quad \Psi''_{\pm} = (\Psi'_{\pm})^2.$$

For a nonnegative solution n , set up the test function

$$\varphi_{\pm} = 2\Psi_{\pm}(n)\Psi'_{\pm}(n)\zeta^2.$$

Then we observe that

$$\begin{aligned} I &= \int_{t_0}^{t_1} \int_{K_{\rho}} n_t \varphi_{\pm} dx dt = \int_{t_0}^{t_1} \int_{K_{\rho}} \frac{d}{dt} [\Psi_{\pm}^2(n)\zeta^2] dx dt \\ &= \int_{K_{\rho} \times \{t_1\}} \Psi_{\pm}^2(n)\zeta^2 dx - \int_{K_{\rho} \times \{t_0\}} \Psi_{\pm}^2(n)\zeta^2 dx. \end{aligned}$$

Using the properties of Ψ_{\pm} , note that

$$\nabla \varphi = 2\nabla n (1 + \Psi_{\pm}(n)) (\Psi'_{\pm}(n))^2 \zeta^2 + 4\Psi_{\pm}(n)\Psi'_{\pm}(n)\zeta \nabla \zeta.$$

Then we calculate various integral quantities:

$$\int_{t_0}^{t_1} \int_{K_{\rho}} \nabla n^{1+\alpha} \nabla \varphi_{\pm} dx dt = II_1 + II_2$$

where

$$\begin{aligned} II_1 &= 2 \int_{t_0}^{t_1} \int_{K_{\rho}} \nabla n^{1+\alpha} \nabla n (1 + \Psi_{\pm}(n)) (\Psi'_{\pm}(n))^2 \zeta^2 dx dt \\ &= 2(1+\alpha) \int_{t_0}^{t_1} \int_{K_{\rho}} n^{\alpha} |\nabla n|^2 (1 + \Psi_{\pm}(n)) (\Psi'_{\pm}(n))^2 \zeta^2 dx dt, \end{aligned}$$

and

$$\begin{aligned} II_2 &= 4 \int_{t_0}^{t_1} \int_{K_{\rho}} \nabla n^{1+\alpha} \Psi_{\pm}(n) \Psi'_{\pm}(n) \zeta \nabla \zeta dx dt \\ &\leq \epsilon_0 (1+\alpha) \int_{t_0}^{t_1} \int_{K_{\rho}} n^{\alpha} |\nabla n|^2 |\Psi'_{\pm}(n)|^2 \zeta^2 dx dt \\ &\quad + 16\epsilon_0^{-1} (1+\alpha) \int_{t_0}^{t_1} \int_{K_{\rho}} n^{\alpha} \Psi_{\pm}^2(n) |\nabla \zeta|^2 dx dt \\ &= II_{21} + II_{22} \end{aligned}$$

applying the Cauchy-Schwartz inequality with any $\epsilon_0 > 0$. By fixing $\epsilon_0 = 2$, the first integral term is absorbed by II_1 .

Next, we handle the integral quantity carrying the lower order term:

$$(7.1) \quad \int_{t_0}^{t_1} \int_{K_\rho} \nabla(Bn)\varphi \, dx \, dt = III_1 + III_2$$

using ∇B . The integral (7.1) produces two terms

$$III_1 = \int_{t_0}^{t_1} \int_{K_\rho} \nabla B n \varphi \, dx \, dt \leq 2 \int_{t_0}^{t_1} \int_{K_\rho} |\nabla B| n \Psi_\pm |\Psi'_\pm| \zeta^2 \, dx \, dt$$

and

$$\begin{aligned} III_2 &= \int_{t_0}^{t_1} \int_{K_\rho} B \nabla n \varphi \, dx \, dt = \int_{t_0}^{t_1} \int_{K_\rho} B [\nabla \Psi_\pm^2] \zeta^2 \, dx \, dt \\ &= - \int_{t_0}^{t_1} \int_{K_\rho} \nabla B \Psi_\pm^2 \zeta^2 \, dx \, dt - 2 \int_{t_0}^{t_1} \int_{K_\rho} B \Psi_\pm^2 \zeta \nabla \zeta \, dx \, dt \\ &= III_{21} + III_{22} \end{aligned}$$

using the integration by parts. Then by applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned} III_{22} &\leq 2 \int_{t_0}^{t_1} \int_{K_\rho} k^\alpha \Psi_\pm^2 |\nabla \zeta|^2 \, dx \, dt \\ &\quad + 2 \int_{t_0}^{t_1} \int_{K_\rho} |B|^2 k^{-\alpha} \Psi_\pm^2 \zeta^2 \, dx \, dt = III_{221} + III_{222} \end{aligned}$$

where III_{221} is bounded by II_{22} .

From that $(n - \mu_\pm \pm k)_\pm \leq k$ and (1.3), we have

$$\begin{aligned} III_{222} &\leq 2k^{-\alpha} \left(\ln \frac{1}{\delta} \right)^2 \|B\|_{2\hat{q}_1, 2\hat{q}_2}^2 \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^\pm(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}, \\ III_1 + III_{221} &\leq \left(\left(\ln \frac{1}{\delta} \right)^2 + \frac{\mu_+ \ln \frac{1}{\delta}}{\delta k} \right) \|\nabla B\|_{\hat{q}_1, \hat{q}_2} \left[\int_{t_0}^{t_1} \left[A_{k,\rho}^\pm(t) \right]^{\frac{q_2}{q_1}} dt \right]^{\frac{2(1+\kappa)}{q_2}}. \end{aligned}$$

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